# Functional Wigner representation of quantum dynamics of Bose-Einstein condensate

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We develop a method of simulating the full quantum field dynamics of multi-mode multi-component Bose-Einstein condensates in a trap. We use the truncated Wigner representation to obtain a probabilistic theory that can be sampled. This method produces c-number stochastic equations which may be solved using conventional stochastic methods. The technique is valid for large mode occupation numbers. We give a detailed derivation of methods of functional Wigner representation appropriate for quantum fields. Our approach describes spatial evolution of spinor components and properly accounts for nonlinear losses. Such techniques are applicable to calculating the leading quantum corrections, including effects such as quantum squeezing, entanglement, EPR correlations, and interactions with engineered nonlinear reservoirs. By using a consistent expansion in the inverse density, we are able to explain an inconsistency in the nonlinear loss equations found by earlier authors. © *2013 AIP Publishing LLC*. [http://dx.doi.org/10.1063/1.4801781]

## I. INTRODUCTION

The Wigner representation[1–3] is a convenient and effective method of simulating the dynamics of bosonic quantum fields,[4] including Bose-Einstein condensates (BECs).[5] It works best in the limit of large particle number, where third-order derivative terms in the Wigner-Moyal time-evolution equation can be truncated, and direct diagonalization approaches[6] become computationally impossible. Large particle number usually implies large numbers of field modes with significant population, which makes two-mode variational approaches[7–9] less accurate. This technique has been applied to a number of quantum dynamics problems in both quantum optical[10–12] and BEC systems, including fragmentation,[13–15] dissipative atom transport,[16] dynamically unstable lattice dynamics,[17] dark solitons,[18,19] turbulence,[20,21] decoherence,[22] and squeezing.[23,24] A comparison of theoretical predictions of quantum fluctuations with experiment has generally resulted in excellent agreement, provided the large particle number criterion is met.[12,25]

The truncated Wigner technique is a numerically robust and useful method for BEC simulations. Other methods such as the positive-P representation[26] are known to work better[25] when the truncation approximation breaks down, and there are a number of studies of applicability that compare the truncated Wigner method with the exact positive-P method[27,28] or, where feasible, Bloch-basis approaches. The typical result found is that the truncated Wigner method gives correct results out to a characteristic break time. At this stage, the accumulated errors can lead to large discrepancies in quantum correlations. The method is weakest when dealing with nonlinear quantum tunneling,[29,30] which depends on both long time dynamics and quantum correlations. Within its domain of applicability the technique is remarkably accurate and stable. The overall picture of how this method is related to other techniques for quantum dynamics has been recently reviewed.[31]

The phase-space treatment of multimode problems can be simplified by working with functional mappings to field operators rather than to single-mode operators. This approach was initially

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introduced by Graham and Haken.[32,33] Later it was used in a number of works[5,13,21,34,35] without formally defining the corresponding transformations or accompanying theorems; a more detailed description was given by Polkovnikov.[36] In order to calculate the approximate evolution of the Wigner function of a system numerically, one has to truncate third-order derivative terms,[4,5,37] and project out modes with low occupation numbers. This further complicates the formal description of the method. Recent developments in ultra-cold atomic physics mean that processes such as nonlinear damping, not considered in detail previously, have also become important. Accordingly, much of the mathematical derivation of these techniques is not readily available.

In this paper we present a formal description of the application of the resulting truncated Wigner representation to simulating the multi-mode dynamics of BECs. We successively reduce the problem in its initial form, the master equation for bosonic field operators, to a system of stochastic differential equations, which have significantly lower computational complexity. While there is a price for making the truncation approximation, we emphasize that this is a systematic expansion in a small parameter, $1/N$, where $N$ is the particle number. Such expansions are also relevant to stochastic diagram techniques,[38] which can be used to formally calculate order-by-order behaviour in such equations. Although not treated here in detail, our identities can be applied to parametric interactions, where the truncation approximation has also been applied to EPR and entanglement problems and compared to more rigorous positive-P simulation methods.[27,28]

The purpose of this paper is to provide a more rigorous proof, within the functional analysis formalism, of several identities that are used for these derivations. We focus especially on the problem of nonlinear damping. This is a dominant relaxation mechanism in BEC systems, and is often ignored or (incorrectly) approximated using linear loss terms. We derive the correct Fokker-Planck drift and noise terms for general multicomponent damping using the $1/N$ expansion, which transforms to an expansion in the inverse particle density for quantum fields. Even in the single-component case, the drift term has both a leading (classical) term and a quantum noise correction to the damping. This is needed to predict the loss behaviour correctly, and is important in high-accuracy simulations. Such corrections — both in the drift and noise — are relevant to topics such as EPR correlations, entanglement, and quantum squeezing in the presence of nonlinear reservoirs, a topic of increasing importance in areas ranging from quantum optics and BEC physics to nanomechanical oscillators.[39–41]

We derive the resulting stochastic differential equations from the functional Fokker-Planck equations, and show when the corresponding truncation approximations are applicable. The final equations can be treated using standard computational techniques for solving ordinary and partial stochastic differential equations.[42–44] There are code generator packages and public domain websites with code available for this purpose.[45,46]

## II. QUANTUM FIELDS AND DYNAMICS

In this paper we consider a $C$-component Bose gas in $D$ effective dimensions. The Hamiltonian for this system is expressed in terms of bosonic field creation and annihilation operators $\hat{\Psi}_j^\dagger(\boldsymbol{x})$ and $\hat{\Psi}_j(\boldsymbol{x}), j = 1 \ldots C$, which obey standard bosonic commutation relations

$$[\hat{\Psi}_j, \hat{\Psi}_k'^\dagger] = \delta_{jk}\delta(\boldsymbol{x}' - \boldsymbol{x}). \tag{1}$$

Here $\boldsymbol{x} \in \mathbb{R}^D$ is a $D$-dimensional coordinate vector, we define $\hat{\Psi}_j \equiv \hat{\Psi}_j(\boldsymbol{x})$ and $\hat{\Psi}_k' \equiv \hat{\Psi}_k(\boldsymbol{x}')$ for brevity (the same abbreviation will be used for all functions of coordinates), and $\delta(\boldsymbol{x}' - \boldsymbol{x})$ is a $D$-dimensional Dirac delta function.

### A. Quantized Hamiltonian

The second-quantized Hamiltonian for the system, integrated with a $D$-dimensional volume measure $d\boldsymbol{x}$, is

$$\hat{H} = \int d\boldsymbol{x} \left\{ \hat{\Psi}_j^\dagger K_{jk} \hat{\Psi}_k + \frac{1}{2} \int d\boldsymbol{x}' \hat{\Psi}_j^\dagger \hat{\Psi}_k'^\dagger U_{jk}(\boldsymbol{x}' - \boldsymbol{x}) \hat{\Psi}_j' \hat{\Psi}_k \right\}, \tag{2}$$

where $U_{jk}$ is the two-body scattering potential, and the single-particle Hamiltonian $K_{jk}$ is

$$K_{jk} = \left(-\frac{\hbar^2}{2m}\nabla^2 + \hbar\omega_j + V_j(\boldsymbol{x})\right)\delta_{jk} + \hbar\Omega_{jk}(t). \tag{3}$$

Here $m$ is the atomic mass, $V_j$ is the external trapping potential for spin $j$, $\hbar\omega_j$ is the internal energy of spin $j$, and $\Omega_{jk}$ represents a time-dependent coupling that is used to rotate one spin projection into another.

If we impose a momentum cutoff $k_c$ and only take into account low-energy modes, the non-local scattering potential $U_{jk}(\boldsymbol{x}' - \boldsymbol{x})$ can be replaced by the contact potential $U_{jk}\delta(\boldsymbol{x}' - \boldsymbol{x})$,[47] giving the effective Hamiltonian

$$\hat{H} = \int d\boldsymbol{x} \left\{ \tilde{\Psi}_j^\dagger K_{jk}\tilde{\Psi}_k + \frac{U_{jk}}{2}\tilde{\Psi}_j^\dagger\tilde{\Psi}_k^\dagger\tilde{\Psi}_j\tilde{\Psi}_k \right\}, \tag{4}$$

where $\tilde{\Psi}_j^\dagger$ and $\tilde{\Psi}_j$ are field operators in the new restricted basis of low-energy modes, which is described in detail in Sec. II C. For $s$-wave scattering in three dimensions the coefficient is $U_{jk} = 4\pi\hbar^2 a_{jk}/m$, where $a_{jk}$ is the scattering length. In general, the coefficient must be renormalized depending on the momentum cutoff,[37,48] but the change is small if $dx_i \gg a_{jk}$, where $dx_i$ is the grid step in dimension $i$.

## B. Master equation

The time-evolution of the quantum density matrix $\hat{\rho}$ with particle losses included can be written as a Markovian master equation[49] for the system,

$$\frac{d\hat{\rho}}{dt} = -\frac{i}{\hbar}\left[\hat{H}, \hat{\rho}\right] + \sum_l \kappa_l \int d\boldsymbol{x}\,\mathcal{L}_l\left[\hat{\rho}\right], \tag{5}$$

where $\boldsymbol{l} = (l_1, l_2, \ldots, l_C)$ is a tuple indicating the number of atoms from each component involved in the inelastic interaction that causes the relevant loss. Here we have introduced local Liouville loss terms, that describe $n$-body collisional losses in the Markovian approximation,

$$\mathcal{L}_l\left[\hat{\rho}\right] = 2\hat{O}_l\hat{\rho}\hat{O}_l^\dagger - \hat{O}_l^\dagger\hat{O}_l\hat{\rho} - \hat{\rho}\hat{O}_l^\dagger\hat{O}_l. \tag{6}$$

The reservoir coupling operators $\hat{O}_l$ are products of local field annihilation operators,

$$\hat{O}_l \equiv \hat{O}_l(\tilde{\boldsymbol{\Psi}}) = \prod_{j=1}^{C} \tilde{\Psi}_j^{l_j}(\boldsymbol{x}), \tag{7}$$

describing local $n$-body collision losses where $n = \left(\sum_{j=1}^{C} l_j\right)$. There is an implicit physical assumption that $l_1, \ldots l_C$, particles of internal state quantum number $j = 1, \ldots C$ all collide simultaneously within the volume corresponding to the inverse momentum cutoff, and are removed from the Bose gas.

This is a minimal approach to the complicated issue of particle loss, since it assumes that the reservoir of "lost" particles does not interact with the original Bose gas. The accuracy of this approach depends on such issues as the trapping mechanism. Since, for massive particles, the particle number is conserved in the non-relativistic limit, "lost" particles are simply in a different quantum state. The assumption that these particles don't interact with the original Bose gas is only valid if the trap is state-selective or the collision is highly exothermic, such that the resulting particles are able to move rapidly away.

It is also possible to treat non-Markovian reservoirs within this formalism, by extending the Hamiltonian to include the detailed loss dynamics, but this is not treated in detail in the present paper.

## C. Field operators and restricted basis

It is always necessary in interacting quantum field theory to define a renormalization together with a momentum cutoff. In the case of the truncated Wigner method, this is more important, as the validity of the truncation approximation may depend on it (see Sec. V for more details). We therefore wish to treat this momentum cutoff procedure more carefully, as it has a direct effect on the number of modes, and hence on the validity of the truncation. For each component $j$ we define an orthonormal basis consisting of $\phi_{j,n}(\boldsymbol{x})$, where $\boldsymbol{n} \in \mathbb{B}_j$ is a mode identifier. The orthonormality and completeness conditions for basis functions are, respectively,

$$\int_A \phi_{j,n}^* \phi_{j,m} d\boldsymbol{x} = \delta_{nm},$$

$$\sum_n \phi_{j,n}^* \phi_{j,n}' = \delta(\boldsymbol{x}' - \boldsymbol{x}), \tag{8}$$

where the exact nature of integration area $A$ depends on the basis set. For example, $A$ is the whole space for harmonic oscillator modes, or a box for plane waves. We assume that the integration $\int d\boldsymbol{x}$ is always performed over $A$.

Standard bosonic field operators from (1) can be decomposed as

$$\hat{\Psi}_j = \sum_{n \in \mathbb{B}_j} \phi_{j,n} \hat{a}_{j,n}, \tag{9}$$

where single mode operators $\hat{a}_{j,n}$ obey bosonic commutation relations, the pair $j, \boldsymbol{n}$ serving as a mode identifier. The cutoff mentioned earlier in Sec. II B will result in operating with some fixed subset of each component's basis. Let $\mathbb{M}_j \subseteq \mathbb{B}_j$ be these subsets. Restricted field operators contain only modes from the subset $\mathbb{M}_j$:

$$\tilde{\Psi}_j = \sum_{n \in \mathbb{M}_j} \phi_{j,n} \hat{a}_{j,n}. \tag{10}$$

Formally, these field operators have the functional type $\tilde{\Psi}_j \in \mathbb{F}\mathbb{H}_{\mathbb{M}_j} \equiv (\mathbb{R}^D \to \mathbb{H}_{\mathbb{M}_j})$, where $\mathbb{H}_{\mathbb{M}_j}$ is the Hilbert space of the restricted subset of modes.

Because of the restricted nature of the operator, commutation relations (1) no longer apply. The following ones should be used instead:

$$\left[\tilde{\Psi}_j, \tilde{\Psi}_k'\right] = \left[\tilde{\Psi}_j^\dagger, \tilde{\Psi}_k'^\dagger\right] = 0, \qquad \left[\tilde{\Psi}_j, \tilde{\Psi}_k'^\dagger\right] = \delta_{jk}\delta_{\mathbb{M}_j}(\boldsymbol{x}', \boldsymbol{x}), \tag{11}$$

where $\delta_{\mathbb{M}}$ is a restricted delta-function. The definition is given in Appendix B, in Definition B.2.

## III. FUNCTIONAL WIGNER REPRESENTATION

In this section we introduce and obtain properties of the functional Wigner representation. This will use a number of definitions and results from functional analysis. The relevant mathematical material that is used in this section is defined and its properties derived in Appendix B.

## A. Single-mode Wigner transformation

As a starting point, we recall that the single-mode Wigner transformation of the operator $\hat{A}$ is defined as

$$\mathcal{W}_{\mathrm{sm}}[\hat{A}] = \frac{1}{\pi^2} \int d^2\lambda \exp(-\lambda\alpha^* + \lambda^*\alpha) \mathrm{Tr}\left\{\hat{A}\hat{D}(\lambda, \lambda^*)\right\}, \tag{12}$$

where the displacement operator $\hat{D}(\lambda, \lambda^*) = \exp(\lambda\hat{a}^\dagger - \lambda^*\hat{a})$ was first introduced by Weyl.[50] The detailed description of the single-mode Wigner function $W(\alpha, \alpha^*) \equiv \mathcal{W}_{\mathrm{sm}}[\hat{\rho}]$, analogous to the one provided here, was given by Moyal[2] or, using a notation close to the one in this paper, by later authors.[51–53] In this subsection we will briefly outline these results.

The first theorem provides a way to transform any master equation written in terms of creation and annihilation operators to a partial differential equation for the Wigner function. In the case of the Wigner function, we take $\hat{A} = \hat{\rho}$.

**Theorem 1.** *For any Hilbert-Schmidt operator $\hat{A}$*

$$\mathcal{W}_{\mathrm{sm}}[\hat{a}\hat{A}] = \left(\alpha + \tfrac{1}{2}\tfrac{\partial}{\partial\alpha^*}\right)\mathcal{W}_{\mathrm{sm}}[\hat{A}], \qquad \mathcal{W}_{\mathrm{sm}}[\hat{a}^\dagger\hat{A}] = \left(\alpha^* - \tfrac{1}{2}\tfrac{\partial}{\partial\alpha}\right)\mathcal{W}_{\mathrm{sm}}[\hat{A}],$$

$$\mathcal{W}_{\mathrm{sm}}[\hat{A}\hat{a}] = \left(\alpha - \tfrac{1}{2}\tfrac{\partial}{\partial\alpha^*}\right)\mathcal{W}_{\mathrm{sm}}[\hat{A}], \qquad \mathcal{W}_{\mathrm{sm}}[\hat{A}\hat{a}^\dagger] = \left(\alpha^* + \tfrac{1}{2}\tfrac{\partial}{\partial\alpha}\right)\mathcal{W}_{\mathrm{sm}}[\hat{A}]. \tag{13}$$

This is paired with a second theorem, which helps extract observables (again, expressed in terms of $\hat{a}^\dagger$ and $\hat{a}$) from the Wigner function.

**Theorem 2.** *For any non-negative integers $r_j$, $s_j$*

$$\left\langle \left\{\hat{a}^r(\hat{a}^\dagger)^s\right\}_{\mathrm{sym}}\right\rangle = \int d^2\alpha\,\left(\alpha^r(\alpha^*)^s\right)W(\alpha,\alpha^*), \tag{14}$$

*where $\{\}_{\mathrm{sym}}$ stands for a symmetrically ordered product of operators.*

## B. Definitions of functional operators

In order to specify domains and ranges of discussed functions, functionals and transformations formally, we will employ a special notation. In general, $F \in A \to B \to C$ will denote a function $F$ that depends on two values of types $A$ and $B$, and has a value of type $C$. Note this expression at the same time describes a function that depends on a value of type $A$, and returns a function with a type $B \to C$. The types can be nested, for example, $(A \to B) \to (C \to D)$ denotes a function to function mapping.

We introduce complex functions $\Lambda(\mathbf{x})$, which play the role of the characteristic c-number $\lambda$ in the single-mode case. The important part of the definition is the functional analogue of the displacement operator.

*Definition 3.* Functional displacement operator $\hat{D}_j \in \mathbb{F}_{\mathbb{M}_j} \to \mathbb{H}_{\mathbb{M}_j}$

$$\hat{D}_j[\Lambda, \Lambda^*] = \exp\int d\mathbf{x}\left(\Lambda\tilde{\Psi}_j^\dagger - \Lambda^*\tilde{\Psi}_j\right),$$

where $\mathbb{F}_{\mathbb{M}_j}$, by analogy with $\mathbb{H}_{\mathbb{M}_j}$, is a space of functions that can be decomposed in terms of mode functions from the subset $\mathbb{M}_j$: $\Lambda \equiv \sum_{\mathbf{n}\in\mathbb{M}_j}\phi_{j,\mathbf{n}}\lambda_{j,\mathbf{n}}$.

It is also convenient to define the displacement functional as

*Definition 4.* Displacement functional $D \in \mathbb{F}_{\mathbb{M}_j} \to \mathbb{F}_{\mathbb{M}_j} \to \mathbb{C}$

$$D[\Lambda, \Lambda^*, \Psi, \Psi^*] = \exp\int d\mathbf{x}\left(-\Lambda\Psi^* + \Lambda^*\Psi\right).$$

It can be shown that the functional displacement operator has properties similar to its single-mode equivalent.

*Lemma 5.*

$$\frac{\delta}{\delta\Lambda'}\hat{D}_j[\Lambda, \Lambda^*] = \hat{D}_j[\Lambda, \Lambda^*](\tilde{\Psi}_j'^\dagger + \frac{1}{2}\Lambda'^*) = (\tilde{\Psi}_j'^\dagger - \frac{1}{2}\Lambda'^*)\hat{D}_j[\Lambda, \Lambda^*],$$

$$-\frac{\delta}{\delta\Lambda'^*}\hat{D}_j[\Lambda, \Lambda^*] = \hat{D}_j(\Lambda, \Lambda^*)(\tilde{\Psi}_j' + \frac{1}{2}\Lambda') = (\tilde{\Psi}_j' - \frac{1}{2}\Lambda')\hat{D}_j[\Lambda, \Lambda^*]. \tag{15}$$

*Proof.* Proved using the Baker-Hausdorff theorem and evaluating integrals.    □

## C. Functional Wigner transformation

In this subsection we will extend the single-mode definition (12) to the multimode case, using a functional notation.

*Definition 6.* A multi-component functional Wigner transformation $\mathcal{W} \in \left( \mathbb{R}^D \rightarrow \prod_{j=1}^{C} \mathbb{H}_{\mathbb{M}_j} \right) \rightarrow \prod_{j=1}^{C} \mathbb{F}_{\mathbb{M}_j} \rightarrow \mathbb{C}$ is defined as

$$\mathcal{W}[\hat{A}] = \frac{1}{\pi^{2 \sum |\mathbb{M}_j|}} \int \delta^2 \mathbf{\Lambda} \left( \prod_{j=1}^{C} D[\Lambda_j, \Lambda_j^*, \Psi_j, \Psi_j^*] \right) \mathrm{Tr} \left\{ \hat{A} \prod_{j=1}^{C} \hat{D}_j[\Lambda_j, \Lambda_j^*] \right\},$$

where $\Lambda_j \in \mathbb{F}_{\mathbb{M}_j}$, and $\int \delta^2 \mathbf{\Lambda} \equiv \int \delta^2 \Lambda_1 \ldots \delta^2 \Lambda_C$. The notation $|\mathbb{M}_j|$ stands for the number of elements in the set $\mathbb{M}_j$, so $\sum |\mathbb{M}_j|$ is the total number of modes in all restricted mode subsets. This transforms a coordinate-dependent operator $\hat{A}$ on a restricted subset of a Hilbert space to a functional $(\mathcal{W}[\hat{A}])[\mathbf{\Psi}, \mathbf{\Psi}^*]$.

Next we introduce the Wigner functional, which is a special case of Wigner transformation.

*Definition 7.* The Wigner functional $W \in \prod_{j=1}^{C} \mathbb{F}_{\mathbb{M}_j} \rightarrow \mathbb{R}$ is

$$W[\mathbf{\Psi}, \mathbf{\Psi}^*] \equiv \mathcal{W}[\hat{\rho}] = \frac{1}{\pi^{2 \sum |\mathbb{M}_j|}} \int \delta^2 \mathbf{\Lambda} \left( \prod_{j=1}^{C} D[\Lambda_j, \Lambda_j^*, \Psi_j, \Psi_j^*] \right) \chi_W,$$

where $\chi_W[\mathbf{\Lambda}, \mathbf{\Lambda}^*]$ is the characteristic functional

$$\chi_W[\mathbf{\Lambda}, \mathbf{\Lambda}^*] = \mathrm{Tr} \left\{ \hat{\rho} \prod_{j=1}^{C} \hat{D}_j[\Lambda_j, \Lambda_j^*] \right\}. \tag{16}$$

The Wigner functional has two important properties analogous to the single-mode case. The first one is used to successively transform operator products.

**Theorem 8.** *For any Hilbert-Schmidt operator $\hat{A}$, if $\mathcal{W}[\hat{A}] \equiv (\mathcal{W}[\hat{A}])[\mathbf{\Psi}, \mathbf{\Psi}^*]$, then*

$$\mathcal{W}[\tilde{\Psi}_j \hat{A}] = \left( \Psi_j + \tfrac{1}{2} \tfrac{\delta}{\delta \Psi_j^*} \right) \mathcal{W}[\hat{A}], \qquad \mathcal{W}[\tilde{\Psi}_j^\dagger \hat{A}] = \left( \Psi_j^* - \tfrac{1}{2} \tfrac{\delta}{\delta \Psi_j} \right) \mathcal{W}[\hat{A}],$$

$$\mathcal{W}[\hat{A} \tilde{\Psi}_j] = \left( \Psi_j - \tfrac{1}{2} \tfrac{\delta}{\delta \Psi_j^*} \right) \mathcal{W}[\hat{A}], \qquad \mathcal{W}[\hat{A} \tilde{\Psi}_j^\dagger] = \left( \Psi_j^* + \tfrac{1}{2} \tfrac{\delta}{\partial \Psi_j} \right) \mathcal{W}[\hat{A}]. \tag{17}$$

*Proof.* The proof uses Lemma 5 given above to transform the $\hat{A} \prod_j \hat{D}_j$ product inside the trace, together with Lemma B.9 from Appendix B to integrate by parts (because of the restriction on $\hat{A}$, the traces will be square-integrable[51]), effectively moving the differentials to their intended places.
□

The second property complements the first one, providing a way to obtain expectations of operator products given the Wigner function. Again, it requires a supplementary lemma.

*Lemma 9. For any non-negative integer r and s:*

$$\left\langle \left\{ (\tilde{\Psi}_j')^r (\tilde{\Psi}_j'^\dagger)^s \right\}_{\mathrm{sym}} \right\rangle = \left( \frac{\delta}{\delta \Lambda_j'} \right)^s \left( -\frac{\delta}{\delta \Lambda_j'^*} \right)^r \chi_W[\mathbf{\Lambda}, \mathbf{\Lambda}^*] \Bigg|_{\mathbf{\Lambda} \equiv 0}. \tag{18}$$

*Proof.* The factor corresponding to the *j*th component in the displacement operator can be expanded as

$$\exp \int d\boldsymbol{x} (\Lambda_j \tilde{\Psi}_j^\dagger - \Lambda_j^* \tilde{\Psi}_j) = \sum_{r,s} \frac{1}{r! s!} \left\{ \left( \int d\boldsymbol{x} \, \Lambda_j \tilde{\Psi}_j^\dagger \right)^r \left( -\int d\boldsymbol{x} \, \Lambda_j^* \tilde{\Psi}_j \right)^s \right\}_{\mathrm{sym}}. \tag{19}$$

We can swap functional derivatives with both integration and multiplication by an independent function, so

$$\frac{\delta}{\delta \Lambda'_j} \left( \int d\boldsymbol{x}\, \Lambda_j \tilde{\Psi}_j^\dagger \right)^r = r\, \tilde{\Psi}_j'^\dagger \left( \int d\boldsymbol{x}\, \Lambda_j \tilde{\Psi}_j^\dagger \right)^{r-1}, \tag{20}$$

This is a familiar result for functional derivative of integrals. We note here that it is correct given our restricted functional derivative definitions since $\tilde{\Psi}_j$ can be expanded in our restricted basis set, by definition (10).

The successive application of the differential gives us

$$\left( \frac{\delta}{\delta \Lambda'_j} \right)^r \left( \int d\boldsymbol{x}\, \Lambda_j \tilde{\Psi}_j^\dagger \right)^r = r!\, (\tilde{\Psi}_j'^\dagger)^r. \tag{21}$$

Similarly for the other differential,

$$\left( -\frac{\delta}{\delta \Lambda'^{*}_j} \right)^s \left( -\int d\boldsymbol{x}\, \Lambda_j \tilde{\Psi}_j^\dagger \right)^s = s!\, (\tilde{\Psi}_j'^\dagger)^s. \tag{22}$$

Thus, the differentiation will eliminate all lower order terms in the expansion, and all higher order terms will be eliminated by setting $\Lambda_j \equiv 0$ for every $j$, leaving only one operator product with the required order. $\square$

**Theorem 10.** *For any non-negative integers $r_j$, $s_j$*

$$\left\langle \left\{ \prod_{j=1}^{C} \tilde{\Psi}_j^{r_j} (\tilde{\Psi}_j^\dagger)^{s_j} \right\}_{\mathrm{sym}} \right\rangle = \int \delta^2 \boldsymbol{\Psi} \left( \prod_{j=1}^{C} \Psi_j^{r_j} (\Psi_j^*)^{s_j} \right) W[\boldsymbol{\Psi}, \boldsymbol{\Psi}^*], \tag{23}$$

*where we have used the functional integration $\int \delta^2 \boldsymbol{\Psi}$ from Definition B.6.*

*Proof.* By definition of the Wigner functional, the right-hand side in the above equation can be written,

$$I = \int \delta^2 \boldsymbol{\Psi} \left( \prod_{j=1}^{C} \Psi_j^{r_j} (\Psi_j^*)^{s_j} \right) W[\boldsymbol{\Psi}, \boldsymbol{\Psi}^*]$$

$$= \frac{1}{\pi^2 \sum |\mathbb{M}_j|} \mathrm{Tr} \left\{ \hat{\rho} \prod_{j=1}^{C} \int \delta^2 \Lambda_j \left( \int \delta^2 \Psi_j\, \Psi_j^{r_j} (\Psi_j^*)^{s_j} D[\Lambda_j, \Lambda_j^*, \Psi_j, \Psi_j^*] \right) \hat{D}_j[\Lambda_j, \Lambda_j^*] \right\}. \tag{24}$$

Evaluating the integral over $\Psi_j$ using Lemma B.8:

$$I = \mathrm{Tr} \left\{ \hat{\rho} \prod_{j=1}^{C} \int \delta^2 \Lambda_j \left( \left( -\frac{\delta}{\delta \Lambda_j^*} \right)^{r_j} \left( \frac{\delta}{\delta \Lambda_j} \right)^{s_j} \Delta_{\mathbb{M}_j}[\Lambda_j] \right) \hat{D}_j[\Lambda_j, \Lambda_j^*] \right\}, \tag{25}$$

where $\Delta_{\mathbb{M}_j}$ is the delta functional from Definition B.7. Integrating by parts for each component, in turn, and eliminating terms which fit Lemma B.10:

$$I = \mathrm{Tr} \left\{ \hat{\rho} \prod_{j=1}^{C} \int \delta^2 \Lambda_j\, \Delta_{\mathbb{M}_j}[\Lambda_j] \left( -\frac{\delta}{\delta \Lambda_j^*} \right)^{r_j} \left( \frac{\delta}{\delta \Lambda_j} \right)^{s_j} \hat{D}_j[\Lambda_j, \Lambda_j^*] \right\}$$

$$= \left( \prod_{j=1}^{C} \left( \frac{\delta}{\delta \Lambda_j} \right)^{s_j} \left( -\frac{\delta}{\delta \Lambda_j^*} \right)^{r_j} \right) \chi_W[\boldsymbol{\Lambda}, \boldsymbol{\Lambda}^*] \Bigg|_{\boldsymbol{\Lambda} \equiv 0}, \tag{26}$$

where $\Delta_{\mathbb{M}_j}$ is a delta functional from Definition B.7. Now, recognizing the final expression as a part of the previous result above in Lemma 9, we immediately get the statement of the theorem. $\square$

## IV. SPECIFIC CASES OF TRANSFORMATIONS

In order to Wigner transform the master equation (5), we will need several theorems about transformations of specific operator products. These theorems employ the expressions for high-order commutators of restricted field operators, which look somewhat similar to those for single-mode bosonic operators, or standard field operators from Ref. 54.

*Lemma 11. Commutators for restricted field operators:*

$$\left[\tilde{\Psi}, (\tilde{\Psi}'^{\dagger})^l\right] = l\delta_{\mathbb{M}}(\boldsymbol{x}', \boldsymbol{x})(\tilde{\Psi}'^{\dagger})^{l-1}, \qquad \left[\tilde{\Psi}^{\dagger}, (\tilde{\Psi}')^l\right] = -l\delta_{\mathbb{M}}^*(\boldsymbol{x}', \boldsymbol{x})(\tilde{\Psi}')^{l-1}. \tag{27}$$

*Proof.* Proved by induction. □

A further generalization of these relations is

*Lemma 12.*

$$\left[\tilde{\Psi}, f(\tilde{\Psi}', \tilde{\Psi}'^{\dagger})\right] = \delta_{\mathbb{M}}(\boldsymbol{x}', \boldsymbol{x})\frac{\partial f}{\partial \tilde{\Psi}'^{\dagger}}, \qquad \left[\tilde{\Psi}^{\dagger}, f(\tilde{\Psi}', \tilde{\Psi}'^{\dagger})\right] = -\delta_{\mathbb{M}}^*(\boldsymbol{x}', \boldsymbol{x})\frac{\partial f}{\partial \tilde{\Psi}'}, \tag{28}$$

*where $f(z, z^*)$ is a function that can be expanded into the power series of $z$ and $z^*$.*

*Proof.* Let us prove the first relation; the procedure for the second one is the same. Without loss of generality, we assume that $f(\tilde{\Psi}', \tilde{\Psi}'^{\dagger})$ can be expanded in power series of normally ordered operators. Using Lemma 11:

$$\begin{aligned}
\left[\tilde{\Psi}, f(\tilde{\Psi}', \tilde{\Psi}'^{\dagger})\right] &= \sum_{r,s} f_{rs}\left[\tilde{\Psi}, (\tilde{\Psi}'^{\dagger})^r(\tilde{\Psi}')^s\right] \\
&= \sum_{r,s} f_{rs}\left[\tilde{\Psi}, (\tilde{\Psi}'^{\dagger})^r\right](\tilde{\Psi}')^s \\
&= \sum_{r,s} f_{rs}\, r\, \delta_{\mathbb{M}}(\boldsymbol{x}', \boldsymbol{x})(\tilde{\Psi}'^{\dagger})^{r-1}(\tilde{\Psi}')^s \\
&= \delta_{\mathbb{M}}(\boldsymbol{x}', \boldsymbol{x})\frac{\partial f}{\partial \tilde{\Psi}'^{\dagger}}.
\end{aligned}$$

□

The simplest case is the transformation of the linear part of the Hamiltonian (2).

**Theorem 13.**

$$\mathcal{W}\left[\left[\int d\boldsymbol{x}\, \tilde{\Psi}_j^{\dagger}\tilde{\Psi}_k, \hat{A}\right]\right] = \int d\boldsymbol{x}\left(-\frac{\delta}{\delta\Psi_j}\Psi_k + \frac{\delta}{\delta\Psi_k^*}\Psi_j^*\right)\mathcal{W}[\hat{A}]. \tag{29}$$

*Proof.* Proved straightforwardly using Theorem 8 and the relation

$$\Psi_k\frac{\delta}{\delta\Psi_j}\mathcal{F} = \left(\frac{\delta}{\delta\Psi_j}\Psi_k - \delta_{jk}\delta_{\mathbb{M}_j}(\boldsymbol{x}, \boldsymbol{x})\right)\mathcal{F}. \tag{30}$$

□

The expression $\delta_{\mathbb{M}_j}(\boldsymbol{x}, \boldsymbol{x})$ will appear in many expressions later in the paper, so we will denote $\tilde{\delta}_j \equiv \delta_{\mathbb{M}_j}(\boldsymbol{x}, \boldsymbol{x})$ for brevity.

Commutators with a Laplacian inside require a somewhat special treatment, because in general the Laplacian acts on basis functions. For our purposes we only need one specific case, and, fortunately, in this case the Laplacian behaves like a constant.

**Theorem 14.**

$$\mathcal{W}\left[\int d\boldsymbol{x}[\tilde{\Psi}^\dagger\nabla^2\tilde{\Psi}, \hat{A}]\right] = \int d\boldsymbol{x}\left(-\frac{\delta}{\delta\Psi}\nabla^2\Psi + \frac{\delta}{\delta\Psi^*}\nabla^2\Psi^*\right)\mathcal{W}[\hat{A}]. \tag{31}$$

*Proof.* Proved using Theorem 8 and Lemma B.11. □

The next theorem describes the transformation of the nonlinear part of the Hamiltonian (4).

**Theorem 15.**

$$\mathcal{W}\left[[\int d\boldsymbol{x}\,\tilde{\Psi}_j^\dagger\tilde{\Psi}_k^\dagger\tilde{\Psi}_j\tilde{\Psi}_k, \hat{A}]\right]$$

$$= \int d\boldsymbol{x}\left(\frac{\delta}{\delta\Psi_j}\left(-\Psi_j\Psi_k\Psi_k^* + \frac{\tilde{\delta}_k}{2}\left(\delta_{jk}\Psi_k + \Psi_j\right)\right)\right.$$

$$+ \frac{\delta}{\delta\Psi_j^*}\left(\Psi_j^*\Psi_k\Psi_k^* - \frac{\tilde{\delta}_k}{2}\left(\delta_{jk}\Psi_k^* + \Psi_j^*\right)\right)$$

$$+ \frac{\delta}{\delta\Psi_k}\left(-\Psi_j\Psi_j^*\Psi_k + \frac{\tilde{\delta}_j}{2}\left(\delta_{jk}\Psi_j + \Psi_k\right)\right)$$

$$+ \frac{\delta}{\delta\Psi_k^*}\left(\Psi_j\Psi_j^*\Psi_k^* - \frac{\tilde{\delta}_j}{2}\left(\delta_{jk}\Psi_j^* + \Psi_k^*\right)\right)$$

$$+ \frac{\delta}{\delta\Psi_j}\frac{\delta}{\delta\Psi_j^*}\frac{\delta}{\delta\Psi_k}\frac{1}{4}\Psi_k - \frac{\delta}{\delta\Psi_j}\frac{\delta}{\delta\Psi_j^*}\frac{\delta}{\delta\Psi_k^*}\frac{1}{4}\Psi_k^*$$

$$\left.+ \frac{\delta}{\delta\Psi_k}\frac{\delta}{\delta\Psi_k^*}\frac{\delta}{\delta\Psi_j}\frac{1}{4}\Psi_j - \frac{\delta}{\delta\Psi_k}\frac{\delta}{\delta\Psi_k^*}\frac{\delta}{\delta\Psi_j^*}\frac{1}{4}\Psi_j^*\right)\mathcal{W}[\hat{A}]. \tag{32}$$

*Proof.* The proof is the same as in the case of Theorem 13. □

Finally, the transformation of loss terms (6) requires some treatment. The proof makes use of two auxiliary lemmas. The first one will help us move functional differentials to their intended places (namely, to the left).

*Lemma 16. For $\mathcal{F} \in \mathbb{F}_\mathbb{M} \to \mathbb{F}$ and any non-negative integer a, b:*

$$\Psi^a\left(\frac{\delta}{\delta\Psi}\right)^b\mathcal{F}[\Psi, \Psi^*]$$

$$= \sum_{j=0}^{\min(a,b)}\binom{b}{j}\frac{(-1)^j a!}{(a-j)!}\tilde{\delta}^j\left(\frac{\delta}{\delta\Psi}\right)^{b-j}\Psi^{a-j}\mathcal{F}[\Psi, \Psi^*] \tag{33}$$

*Proof.* Proved straightforwardly by induction. □

The second lemma gives a way to simplify sums obtained from the application of the previous lemma.

*Lemma 17* (Sum rearrangement). *For any non-negative integer l, u:*

$$\sum_{j=0}^{l}\sum_{k=0}^{\min(l-u,j)} x^{j-k}Q(j,k) = \sum_{v=0}^{l}x^v\sum_{k=0}^{l-\max(u,v)}Q(v+k,k). \tag{34}$$

*Proof.* Obviously, the order $v = j - k$ of factor $f$ can vary from 0 (say, when $j = 0$ and $k = 0$) to $l$ (when $j = l$ and $k = 0$). Therefore.

$$\sum_{j=0}^{l} \sum_{k=0}^{\min(l-u,j)} f^{j-k} g(j,k) = \sum_{v=0}^{l} f^v \sum_{k \in K(l,u,v)} g(v+k,k),$$

where the set $K$ is defined as

$$K(l,u,v) = \{k | 0 \le j \le l \wedge 0 \le k \le \min(l-u,j) \wedge j - k = v\}$$
$$= \{k | k \le l - v \wedge 0 \le k \le \min(l-u, v+k)\}.$$

It is convenient to consider two cases separately $v \le u$ and $v > u$. For the former case

$$K_{v \le u} = \{k | k \le l - v \wedge 0 \le k \le \min(l-u, k+v) \wedge v \le u\}.$$

When $v \le u$, $k \le l - v \le l - u \le \min(l-u, k+v)$ is always true, and the first inequation is redundant,

$$K_{v \le u} = \{k | 0 \le k \le \min(l-u, v+k) \wedge v \le u\}.$$

Splitting into two sets to get rid of the minimum function:

$$K_{v \le u} = \{k | v \le u \wedge k \ge 0 \wedge ((k \le l-u \wedge l-u < v+k) \vee (k \le v+k \wedge l-u \ge v+k))\}$$
$$= \{k | v \le u \wedge 0 \le k \le l-u\}.$$

For the latter case,

$$K_{v > u} = \{k | k \le l-v \wedge 0 \le k \le \min(l-u, k+v) \wedge v > u\}$$
$$= \{k | v > u \wedge k \ge 0 \wedge ((k \le l-v \wedge k \le l-u \wedge l-u \le k+v)$$
$$\vee (k \le l-v \wedge k \le k+v \wedge l-u > k+v))\}$$
$$= \{k | v > u \wedge 0 \le k \le l-v\}.$$

Thus

$$K = K_{v \le u} \cup K_{v > u}$$
$$= \{k | v \le u \wedge 0 \le k \le l-u\} \cup \{k | v > u \wedge 0 \le k \le l-v\}$$
$$= \{k | 0 \le k \le l - \max(u,v)\},$$

which gives us the statement of the lemma.                                                             □

Finally, the loss transformation theorem can be proved.

**Theorem 18.** *The Wigner transformation of loss term (6) is*

$$\mathcal{W}\left[\int d\boldsymbol{x}\, \mathcal{L}_l[\hat{A}]\right] = \int d\boldsymbol{x} \sum_{j_1=0}^{l_1} \sum_{k_1=0}^{l_1} \cdots \sum_{j_C=0}^{l_C} \sum_{k_C=0}^{l_C} \left(\prod_{c=1}^{C} \left(\frac{\delta}{\delta \Psi_c^*}\right)^{j_c} \left(\frac{\delta}{\delta \Psi_c}\right)^{k_c}\right) L_{l,j,k} \mathcal{W}[\hat{A}], \quad (35)$$

*where the nonlinear loss coefficient L is*

$$L_{l,j,k} = \left(2 - (-1)^{\sum_c j_c} - (-1)^{\sum_c k_c}\right) \times$$

$$\times \prod_{c=1}^{C} \left(\sum_{m_c=0}^{l_c - \max(j_c, k_c)} Q(l_c, j_c, k_c, m_c) \delta_{\mathbb{M}_c}^{m_c}(\boldsymbol{x}, \boldsymbol{x}) \Psi_c^{l_c - j_c - m_c} (\Psi_c^*)^{l_c - k_c - m_c}\right), \quad (36)$$

*and we introduce a numerical factor Q, where*

$$Q(l, j, k, m) = \frac{(-1)^m}{2^{j+k+m}} \frac{(l!)^2}{m!\, j!\, k!\, (l-k-m)!\, (l-j-m)!}. \quad (37)$$

*Proof.* Proved by applying Theorem 8, expanding products using binomial theorem, using Lemma 16 to move differentials to front, and applying Lemma 17 to transform the resulting summations.     □

## V. WIGNER TRUNCATION AND FOKKER-PLANCK EQUATION

Now we have all necessary tools to transform the master equation (5) with the Wigner transformation from Definition 6 to the form of a partial differential equation.

The single-particle term (3) is transformed using Theorems 13 and 14 (since $K_j$ is basically a sum of Laplacian operator and functions of $\boldsymbol{x}$):

$$\mathcal{W}\left[ [\int d\boldsymbol{x}\, \tilde{\Psi}_j^\dagger K_{jk} \tilde{\Psi}_k, \hat{\rho}] \right] = \int d\boldsymbol{x} \left( -\frac{\delta}{\delta \Psi_j} K_{jk} \Psi_k + \frac{\delta}{\delta \Psi_k^*} K_{jk} \Psi_j^* \right) W, \tag{38}$$

where the Wigner function $W \equiv \mathcal{W}[\hat{\rho}]$. The nonlinear term is transformed with Theorem 15 (assuming $U_{kj} = U_{jk}$):

$$\mathcal{W}\left[ [\int d\boldsymbol{x}\, \frac{U_{jk}}{2} \tilde{\Psi}_j^\dagger \tilde{\Psi}_k^\dagger \tilde{\Psi}_j \tilde{\Psi}_k, \hat{\rho}] \right] = \int d\boldsymbol{x}\, U_{jk} \left( \frac{\delta}{\delta \Psi_j} \left( -\Psi_j \Psi_k \Psi_k^* + \frac{\tilde{\delta}_k}{2}(\delta_{jk} \Psi_k + \Psi_j) \right) \right.$$
$$+ \frac{\delta}{\delta \Psi_j^*} \left( \Psi_j^* \Psi_k \Psi_k^* - \frac{\tilde{\delta}_k}{2}(\delta_{jk} \Psi_k^* + \Psi_j^*) \right)$$
$$\left. + \frac{\delta}{\delta \Psi_j} \frac{\delta}{\delta \Psi_j^*} \frac{\delta}{\delta \Psi_k} \frac{1}{4} \Psi_k - \frac{\delta}{\delta \Psi_j} \frac{\delta}{\delta \Psi_j^*} \frac{\delta}{\delta \Psi_k^*} \frac{1}{4} \Psi_k^* \right) W. \tag{39}$$

Loss terms (6) are transformed with Theorem 18 and result in a similar equation, with a finite number of differential terms up to order $2n$ for $n$ −body collisional losses. It should be recalled that in the above equation, the notation $\tilde{\delta}_j \equiv \delta_{\mathbb{M}_j}(\boldsymbol{x}, \boldsymbol{x})$ was introduced previously for brevity; in general this is a cut-off dependent constant with units of density.

Assuming that $K_{jk}$, $U_{jk}$, and $\kappa_l$ are real-valued, all the transformations described above result in a partial differential equation for $W$ of the form

$$\frac{\partial W}{\partial t} = \int d\boldsymbol{x} \left\{ -\sum_{j=1}^C \frac{\delta}{\delta \Psi_j} \mathcal{A}_j - \sum_{j=1}^C \frac{\delta}{\delta \Psi_j^*} \mathcal{A}_j^* + \sum_{j=1}^C \sum_{k=1}^C \frac{\delta^2}{\delta \Psi_j^* \delta \Psi_k} \mathcal{D}_{jk} + \mathrm{O}\left[ \frac{\delta^3}{\delta \Psi_j^3} \right] \right\} W. \tag{40}$$

Terms of order higher than 2 are produced both by the nonlinear term in the Hamiltonian and loss terms. Such an equation could be solved perturbatively if there were only orders up to 3 (which means an absence of nonlinear losses),[55] but in most cases all terms except for first- and second-order ones are truncated. In order to justify this truncation in a consistent way, we develop an order-by-order expansion in $1/N_c$, where $N_c$ is a characteristic particle number in a physical interaction volume, and truncate terms of formal order $1/N_c^2$. This is achieved[4] by use of the formal definition of a scaled Wigner function $W^\psi$, satisfying a scaled equation in terms of dimensionless scaled fields $\psi$, with

$$\psi_j = \Psi_j \sqrt{\ell_c^D/N_c}$$
$$\mathcal{A}_j^\psi = t_c \sqrt{\ell_c^D/N_c}\, \mathcal{A}_j + \mathrm{O}\left(1/N_c^2\right)$$
$$\mathcal{D}_{jk}^\psi = t_c \left(\ell_c^D/N_c\right) \mathcal{D}_{jk} + \mathrm{O}\left(1/N_c^2\right). \tag{41}$$

Here $t_c$ is a characteristic interaction time and $\ell_c$ is a characteristic interaction length. These would normally be chosen as the healing time and healing length, respectively, in a BEC calculation. Typically, the cell size is chosen as proportional to the healing length, for optimum accuracy in resolving spatial detail. Using this expansion, a consistent order-by-order expansion in $(1/N_c)$ can

be obtained, of form

$$\frac{\partial W^\psi}{\partial \tau} = \int d\boldsymbol{x} \left\{ -\sum_{j=1}^{C} \frac{\delta}{\delta \psi_j} \mathcal{A}_j^\psi - \sum_{j=1}^{C} \frac{\delta}{\delta \psi_j^*} \mathcal{A}_j^{\psi *} + \sum_{j=1}^{C} \sum_{k=1}^{C} \frac{\delta^2}{\delta \psi_j^* \delta \psi_k} \mathcal{D}_{jk}^\psi + \mathrm{O}\left[\frac{1}{N_c^2}\right] \right\} W^\psi. \quad (42)$$

With the assumption of the state being coherent, the simple condition for truncation — i.e., omitting terms of $O(1/N_c^2)$ — can be shown to be[37]

$$N_j \gg |\mathbb{M}_j|, \quad (43)$$

where $N_j$ is the total number of atoms of the component $j$. The inclusion of the mode factor is caused by the fact that the number of additional terms increases as the number of modes increases, which may be needed to treat convergence of the method for large momentum cutoff. We see immediately that there are subtleties involved if one wishes to include larger numbers of high-momentum modes, since this increases the mode number while leaving the numbers unchanged. In other words, the truncation technique is inherently restricted in its ability to resolve fine spatial details in the high-momentum cutoff limit.

The $1/N_c$ is equivalent to an expansion in the inverse density, which requires the inequality[56]

$$\tilde{\delta}_j = \delta_{\mathbb{M}_j}(\boldsymbol{x}, \boldsymbol{x}) \ll |\Psi_j|^2. \quad (44)$$

The coherency assumption does not, of course, encompass all possible states that can be produced during evolution, which means that the condition above is more of a guide than a restriction. For certain systems the truncation was shown to work even when (44) is violated.[16] The validity may also depend on the simulation time,[57] and other physically relevant factors.

A common example of such relevant factors is that there can be a large difference in the size of the original parameters. To illustrate this issue, one may have a situation where $\kappa_1 \approx \kappa_2 N_c$ even though $N_c \gg 1$. Under these conditions, it is essential to include a scaling of the parameters in calculating the formal order, so that the scaled parameters have comparable sizes. This allows one to correctly identify which terms are negligible in a given physical problem, and which terms must be included.

In general, one can estimate the validity of truncation for the particular problem and the particular observable by calculating the quantum correction.[36] Other techniques for estimating validity include comparison with the exact positive-P simulation method,[4] and examining results for unphysical behaviour such as negative occupation numbers.[25] It is generally the case for unitary evolution that errors caused by truncation grow in time, leading to a finite time horizon for applicability, as explained in the Introduction.

The use of this Wigner truncation allows us to simplify the results of Theorems 15 and 18. Wigner truncation is an expansion up to the order $1/N_c$, so during the simplification, along with the higher order derivatives, we drop all components with $\tilde{\delta}_j$ of order higher than 1 in the drift terms, and of order higher than 0 in the diffusion terms.

*Lemma 19. Assuming the conditions for Wigner truncation are satisfied, the result of Wigner transformation of the nonlinear term can be written as*

$$\mathcal{W}\left[ \left[\frac{U_{jk}}{2} \tilde{\Psi}_j^\dagger \tilde{\Psi}_k^\dagger \tilde{\Psi}_j \tilde{\Psi}_k, \hat{\rho}\right] \right] \approx U_{jk} \left( \frac{\delta}{\delta \Psi_j} \left( -\Psi_j \Psi_k \Psi_k^* + \frac{\tilde{\delta}_k}{2} (\delta_{jk} \Psi_k + \Psi_j) \right) \right.$$
$$\left. + \frac{\delta}{\delta \Psi_j^*} \left( \Psi_j^* \Psi_k \Psi_k^* - \frac{\tilde{\delta}_k}{2} (\delta_{jk} \Psi_k^* + \Psi_j^*) \right) \right) W.$$

*Proof.* Proved by simplifying Eq. (39) under the Wigner truncation condition (essentially by dropping terms with third order derivatives). ∎

*Lemma 20. Assuming the conditions for Wigner truncation are satisfied, the result of Wigner transformation of the loss term can be written as*

$$
\mathcal{W}[\mathcal{L}_l[\hat{\rho}]] \approx \left( \sum_{j=1}^{C} \frac{\delta}{\delta \Psi_j^*} \left( \frac{\partial O_l}{\partial \Psi_j} O_l^* - \frac{1}{2} \sum_{k=1}^{C} \tilde{\delta}_k \frac{\partial^2 O_l}{\partial \Psi_j \partial \Psi_k} \frac{\partial O_l^*}{\partial \Psi_k^*} \right) \right.
$$
$$
+ \sum_{j=1}^{C} \frac{\delta}{\delta \Psi_j} \left( \frac{\partial O_l^*}{\partial \Psi_j^*} O_l - \frac{1}{2} \sum_{k=1}^{C} \tilde{\delta}_k \frac{\partial^2 O_l^*}{\partial \Psi_j^* \partial \Psi_k^*} \frac{\partial O_l}{\partial \Psi_k} \right)
$$
$$
\left. + \sum_{j=1}^{C} \sum_{k=1}^{C} \frac{\delta^2}{\delta \Psi_j^* \delta \Psi_k} \frac{\partial O_l}{\partial \Psi_j} \frac{\partial O_l^*}{\partial \Psi_k^*} \right) W, \tag{45}
$$

*where* $O_l \equiv O_l[\mathbf{\Psi}] = \prod_{j=1}^{C} \Psi_j^{l_j}$.

*Proof.* The proof is basically a simplification of the result of Theorem 18 under two conditions. First, we neglect all terms with order lower than $1/N$. This means that we are only considering terms with $\sum m_c \leq 1$ in the drift part, and $\sum m_c = 0$ in the diffusion part. Second, we are dropping all terms with high order differentials, which can be expressed as limiting $\sum j_c + \sum k_c \leq 2$. The only combinations of $j_c$ and $k_c$ for which $Z(\mathbf{j}, \mathbf{k})$ is not zero are thus $\{j_c = \delta_{cn}, k_c = 0, n \in [1, C]\}$, $\{j_c = 0, k_c = \delta_{cn}, n \in [1, C]\}$ and $\{j_c = \delta_{cn}, k_c = \delta_{cp}, n \in [1, C], p \in [1, C]\}$. These combinations produce terms with $\delta/\delta \Psi_n^*$, $\delta/\delta \Psi_n$ (drift) and $\delta^2/\delta \Psi_p \delta \Psi_n^*$ (diffusion), respectively. Applying these conditions one can get the statement of the theorem. $\square$

Thus the truncated Fokker-Planck equation (FPE) is

$$
\frac{dW}{dt} = \int d\mathbf{x} \left( -\sum_{j=1}^{C} \frac{\delta}{\delta \Psi_j} \mathcal{A}_j - \sum_{j=1}^{C} \frac{\delta}{\delta \Psi_j^*} \mathcal{A}_j^* + \sum_{j=1}^{C} \sum_{k=1}^{C} \frac{\delta^2}{\delta \Psi_j^* \delta \Psi_k} \mathcal{D}_{jk} \right) W, \tag{46}
$$

or, in matrix form

$$
\frac{dW}{dt} = \int d\mathbf{x} \left( -2\mathrm{Re}\left( \boldsymbol{\delta}_{\mathbf{\Psi}} \cdot \boldsymbol{\mathcal{A}} \right) + \mathrm{Tr}\left\{ \boldsymbol{\delta}_{\mathbf{\Psi}*} \boldsymbol{\delta}_{\mathbf{\Psi}}^T \mathcal{D} \right\} \right) W,
$$

where we define the relevant coefficients in the FPE as

$$
\mathcal{A}_j = -\frac{i}{\hbar} \left( \sum_{k=1}^{C} K_{jk} \Psi_k + \Psi_j \sum_{k=1}^{C} U_{jk} \left( |\Psi_k|^2 - \frac{\delta_{jk}+1}{2} \tilde{\delta}_k \right) \right)
$$
$$
- \sum_l \kappa_l \left( \frac{\partial O_l^*}{\partial \Psi_j^*} O_l - \frac{1}{2} \sum_{k=1}^{C} \tilde{\delta}_k \frac{\partial^2 O_l^*}{\partial \Psi_j^* \partial \Psi_k^*} \frac{\partial O_l}{\partial \Psi_k} \right), \tag{47}
$$

and

$$
\mathcal{D}_{jk} = \sum_l \kappa_l \frac{\partial O_l}{\partial \Psi_j} \frac{\partial O_l^*}{\partial \Psi_k^*}. \tag{48}
$$

## VI. STOCHASTIC DIFFERENTIAL EQUATIONS

Direct solution of the above FPE is generally impractical, and a Monte Carlo or sampled calculation is called for. Since the diffusion matrix is positive-definite, the truncated Wigner function $W$ is a probability distribution, provided it has a positive initial distribution. Therefore, the equation can be further transformed to the equivalent set of stochastic differential equations in Itô form.

## A. Stochastic evolution

General results on such transformations are given in Appendix C, as described by Theorem C.4. Application of these methods to the truncated FPE (46) gives immediately a system of SDEs in Itô[58] form,

$$d\Psi = \mathcal{P}\left[\mathcal{A}dt + \mathcal{B}d\boldsymbol{Q}\right],$$  (49)

where the drift term $\mathcal{A}$ is given by (47), and noise term is a matrix with elements

$$\mathcal{B}_{jl} = \sqrt{\kappa_l}\frac{\partial O_l^*}{\partial \Psi_j^*}.$$  (50)

Here $Q_l$ is a functional Wiener process,

$$Q_l = \sum_{\boldsymbol{n}\in\mathbb{B}}\phi_j Z_{l,\boldsymbol{n}},$$  (51)

and $Z_{l,\boldsymbol{n}}$ are, in turn, independent complex-valued Wiener processes with $\langle Z_{l,\boldsymbol{n}}Z_{k,\boldsymbol{m}}^*\rangle = \delta_{l,k}\delta_{\boldsymbol{n},\boldsymbol{m}}dt$.

Alternatively, in Stratonovich form the SDEs look like

$$d\Psi = \mathcal{P}\left[(\mathcal{A}-\mathcal{S})dt + \mathcal{B}d\boldsymbol{Q}\right],$$  (52)

where the Stratonovich[58] term has components

$$\mathcal{S}_j = \frac{1}{2}\sum_{n=1}^{C}\sum_{l}\kappa_l\frac{\partial O_l}{\partial \Psi_n}\left(\frac{\partial^2 O_l}{\partial \Psi_n \partial \Psi_j}\right)^*\delta_{\mathbb{M}_n}(\boldsymbol{x},\boldsymbol{x}).$$  (53)

These equations can now be solved using conventional methods,[35] and any required expectations of symmetrically ordered operator products can be obtained from their solution using Theorem 10:

$$\left\langle\left\{\prod_{j=1}^{C}\tilde{\Psi}_j^{r_j}(\tilde{\Psi}_j^\dagger)^{s_j}\right\}_{\text{sym}}\right\rangle = \int \delta\Psi\left(\prod_{j=1}^{C}\Psi_j^{r_j}(\Psi_j^*)^{s_j}\right)W$$

$$\approx \left\langle\prod_{j=1}^{C}\Psi_j^{r_j}(\Psi_j^*)^{s_j}\right\rangle_{\text{paths}},$$  (54)

where $r_c$ and $s_c$ is some set of non-negative integers, and $\langle\rangle_{\text{paths}}$ stands for the average over the simulation paths.

## B. Single-component example

To illustrate the application of the theorems above to some specific problems we will first consider a simple case with a single component BEC, with 3-body loss and no unitary evolution (the same as described by Norrie et al.[21]). For this system we have $\hat{K}\equiv 0$, $U\equiv 0$, and $\hat{O}=\tilde{\Psi}^3$ (and, consequently, $O=\Psi^3$), and we also denote $\gamma = 6\kappa$. The FPE for this system is therefore

$$\frac{dW}{dt} = -\frac{\delta}{\delta\Psi}\left(-\frac{\gamma}{2}|\Psi|^4\Psi + \frac{3\gamma}{2}|\Psi|^2\Psi\tilde{\delta} - \frac{3\gamma}{4}\Psi\tilde{\delta}^2\right)$$

$$-\frac{\delta}{\delta\Psi^*}\left(-\frac{\gamma}{2}|\Psi|^4\Psi^* + \frac{3\gamma}{2}|\Psi|^2\Psi^*\tilde{\delta} - \frac{3\gamma}{4}\Psi^*\tilde{\delta}^2\right)$$

$$+\frac{\delta^2}{\delta\Psi^*\delta\Psi}\left(\frac{3\gamma}{2}|\Psi|^4 - 3\gamma|\Psi|^2|\tilde{\delta} + \frac{3\gamma}{4}\tilde{\delta}^2\right)$$

$$+\frac{\delta^3}{\delta\Psi^*\delta\Psi^2}\left(\frac{3\gamma}{8}|\Psi|^2\Psi - \frac{3\gamma}{8}\Psi\tilde{\delta}\right) + \frac{\delta^3}{\delta\Psi^{*2}\delta\Psi}\left(\frac{3\gamma}{8}|\Psi|^2\Psi^* - \frac{3\gamma}{8}\Psi^*\tilde{\delta}\right)$$

$$+\frac{\delta}{\delta\Psi^3}\left(\frac{\gamma}{24}\Psi^3\right) + \frac{\delta^3}{\delta\Psi^{*3}}\left(\frac{\gamma}{24}\Psi^{*3}\right) + \text{O}[\frac{1}{N_c^4}].$$

After the truncation, the resulting stochastic equation describing the system is

$$d\Psi = \mathcal{P}\left[ -\frac{\gamma}{6}\left( \frac{\partial O^*}{\partial \Psi^*}O - \frac{1}{2}\tilde{\delta}\frac{\partial^2 O^*}{\partial(\Psi^*)^2}\frac{\partial O}{\partial \Psi} \right)dt + \sqrt{\frac{\gamma}{6}}\frac{\partial O^*}{\partial \Psi^*}dQ(\boldsymbol{x},t) \right]$$

$$= \mathcal{P}\left[ -\left( \frac{\gamma}{2}|\Psi|^4\Psi - \frac{3\gamma}{2}\tilde{\delta}|\Psi|^2\Psi \right)dt + \sqrt{\frac{3\gamma}{2}}(\Psi^*)^2 dQ(\boldsymbol{x},t) \right].$$

The equation coincides with the one given by Norrie $et\ al.$,[21] except for the additional correction to the drift term, which is of order $1/N$ and therefore cannot be omitted.

If we calculate the rate population change over time using Itô formula (either by expanding $\Psi$ in mode form, or using the functional equivalent of Itô formula), we obtain

$$\frac{dN}{dt} = \frac{d\langle\tilde{\Psi}^\dagger\tilde{\Psi}\rangle}{dt} = \frac{d\langle\Psi^*\Psi\rangle_{\text{paths}}}{dt} = -\gamma\int d\boldsymbol{x}\left( \langle|\Psi|^6\rangle_{\text{paths}} - \frac{9}{2}\tilde{\delta}\langle|\Psi|^4\rangle_{\text{paths}} \right).$$

This can be transformed further to more conventional form. Using the equivalence (54) and the analogue of the ordering transformation formula[51] for field operators

$$\left\{ (\tilde{\Psi}^\dagger)^r\,\tilde{\Psi}^s \right\}_{\text{sym}} = \sum_{k=0}^{\min(r,s)} \frac{k!}{2^k}\binom{r}{k}\binom{s}{k}(\tilde{\Psi}^\dagger)^{r-k}\tilde{\Psi}^{s-k}\tilde{\delta}^k,$$

we get

$$\langle|\Psi|^4\rangle_{\text{paths}} = g^{(2)}n^2 + 2\tilde{\delta}n + \frac{1}{2}\tilde{\delta}^2,$$

$$\langle|\Psi|^6\rangle_W = g^{(3)}n^3 + \frac{9}{2}\tilde{\delta}g^{(2)}n^2 + \frac{9}{2}\tilde{\delta}^2 n + \frac{3}{4}\tilde{\delta}^3.$$

Here $n = \langle\tilde{\Psi}^\dagger\tilde{\Psi}\rangle$ is the particle density, and $g^{(k)} = \langle(\tilde{\Psi}^\dagger)^k\tilde{\Psi}^k\rangle/\langle\tilde{\Psi}^\dagger\tilde{\Psi}\rangle$ are correlation factors. Substituting above expressions into the equation for the population rate:

$$\frac{dN}{dt} = -\gamma\int d\boldsymbol{x}\left( g^{(3)}n^3 - \frac{9}{2}\tilde{\delta}^2 n - \frac{3}{2}\tilde{\delta}^3 \right).$$

We see that the second highest term in the expression is canceled, which agrees with the expansion being correct up to the order $1/N$. If the quantum correction term to the drift is omitted, one finds that a physically incorrect quadratic nonlinear term proportional to $n^2$ is obtained, which is inconsistent with an exact short-time solution to the master equation.[21]

### C. Two-component example

As a more involved example, let us consider a two component $^{87}$Rb BEC from recent experiments.[22,23] In this case we have both unitary evolution (including nonlinear interaction) (4), and three sources of losses: three-body recombination $\hat{O}_{111} = \tilde{\Psi}_1^3$, two-body interspecies loss $\hat{O}_{12} = \tilde{\Psi}_1\tilde{\Psi}_2$ and two-body intraspecies loss $\hat{O}_{22} = \tilde{\Psi}_2^2$. This gives us stochastic differential equations (SDEs) (49) with drift terms

$$\mathcal{A}_1 = -\frac{i}{\hbar}\left( \sum_{k=1}^{2}K_{1k}\Psi_k + \Psi_1\sum_{k=1}^{2}U_{1k}\left( |\Psi_k|^2 - \frac{\delta_{1k}+1}{2}\tilde{\delta}_k \right) \right)$$

$$- 3\kappa_{111}\left( |\Psi_1|^2 - 3\tilde{\delta}_1 \right)|\Psi_1|^2\Psi_1 - \kappa_{12}\left( |\Psi_2|^2 - \frac{\tilde{\delta}_2}{2} \right)\Psi_1,$$

$$\mathcal{A}_2 = -\frac{i}{\hbar}\left( \sum_{k=1}^{2}K_{2k}\Psi_k + \Psi_2\sum_{k=1}^{C}U_{2k}\left( |\Psi_k|^2 - \frac{\delta_{2k}+1}{2}\tilde{\delta}_k \right) \right)$$

$$- \kappa_{12}\left( |\Psi_1|^2 - \frac{\tilde{\delta}_1}{2} \right)\Psi_2 - 2\kappa_{22}\left( |\Psi_2|^2 - \tilde{\delta}_2 \right)\Psi_2.$$

and noise terms

$$\mathcal{B}_{1,111} = 3\sqrt{\kappa_{111}} \left(\Psi_1^*\right)^2, \quad \mathcal{B}_{1,12} = \sqrt{\kappa_{12}}\Psi_2^*, \quad \mathcal{B}_{1,22} = 0,$$

$$\mathcal{B}_{2,111} = 0, \quad \mathcal{B}_{2,12} = \sqrt{\kappa_{12}}\Psi_1^*, \quad \mathcal{B}_{2,22} = 2\sqrt{\kappa_{22}}\Psi_2^*.$$

This type of stochastic equation is needed to treat coherent BEC interferometry in the presence of nonlinear loss terms caused by two and three body collisions.

## D. Initial states

Initial values for the numerical integration of Eq. (49) are obtained by finding the Wigner transformation of the density matrix for the desired initial state, and then sampling the initial values according to the resulting Wigner function. As an example of the procedure, consider the simple case with a single-component coherent initial state.

**Theorem 21.** *The Wigner distribution for a multi-mode coherent state with the expectation value* $\Psi^{(0)} \equiv \sum_{n \in \mathbb{M}} \alpha_n^{(0)}\phi_n$ *is*

$$W_c[\Psi, \Psi^*] = \left(\frac{2}{\pi}\right)^{|\mathbb{M}|} \prod_{n \in \mathbb{M}} \exp(-2|\alpha_n - \alpha_n^{(0)}|^2), \tag{55}$$

*where* $\Psi \equiv \sum_{n \in \mathbb{M}} \alpha_n \phi_n$.

*Proof.* The density matrix of the state is

$$\hat{\rho} = |\alpha_n^{(0)}, \, \boldsymbol{n} \in \mathbb{M}\rangle\langle\alpha_n^{(0)}, \, \boldsymbol{n} \in \mathbb{M}| = \left(\prod_{n \in \mathbb{M}} |\alpha_n^{(0)}\rangle\right) \left(\prod_{n \in \mathbb{M}} \langle\alpha_n^{(0)}|\right). \tag{56}$$

Then the characteristic functional for this state can be expressed as

$$\chi_W[\Lambda, \Lambda^*] = \prod_{n \in \mathbb{M}} \langle\alpha_n^{(0)}|\hat{D}_n(\lambda_n, \lambda_n^*)|\alpha_n^{(0)}\rangle, \tag{57}$$

where $\lambda_n$ are coefficients in the decomposition of $\Lambda \in \mathbb{F}_{\mathbb{M}}$. Using the properties of the displacement operator, this can be transformed to

$$\chi_W[\Lambda, \Lambda^*] = \prod_{n \in \mathbb{M}} \exp(-\lambda_n^* \alpha_n^{(0)} + \lambda_n (\alpha_n^{(0)})^* - \frac{1}{2}|\lambda|^2). \tag{58}$$

Finally, the Wigner function is

$$W_c[\Psi, \Psi^*] = \frac{1}{\pi^{2|\mathbb{M}|}} \prod_{n \in \mathbb{M}} \left(\int d^2\lambda_n \exp(-\lambda_n(\alpha_n^* - (\alpha_n^{(0)})^*) + \lambda_n^*(\alpha_n - \alpha_n^{(0)}) - \frac{1}{2}|\lambda|^2)\right)$$

$$= \left(\frac{2}{\pi}\right)^{|\mathbb{M}|} \prod_{n \in \mathbb{M}} \exp(-2|\alpha_n - \alpha_n^{(0)}|^2).$$

$$\square$$

The resulting Wigner distribution is a product of independent complex-valued Gaussian distributions for each mode, with an expectation value equal to the expectation value of the mode, and variance equal to $\frac{1}{2}$. Therefore, the initial state can be sampled as

$$\alpha_n = \alpha_n^{(0)} + \frac{1}{\sqrt{2}}\eta_n, \tag{59}$$

where $\eta_n$ are normally distributed complex random numbers with zero mean, $\langle\eta_m\eta_n\rangle = 0$ and $\langle\eta_m\eta_n^*\rangle = \delta_{m,n}$ or, in other words, with real components distributed independently with variance $\frac{1}{2}$.

This looks like adding half a "vacuum particle" to each mode. In functional form this can be written as

$$\Psi(\boldsymbol{x}, 0) = \Psi^{(0)}(\boldsymbol{x}, 0) + \sum_{\boldsymbol{n} \in \mathbb{M}} \frac{\eta_{\boldsymbol{n}}}{\sqrt{2}} \phi_{\boldsymbol{n}},$$

where $\Psi^{(0)}(\boldsymbol{x}, 0)$ is the "classical" ground state of the system.

More involved examples, including thermalized states and Bogoliubov states, are reviewed by Blakie *et al.*,[35] and Ruostekoski and Martin.[59] In particular, a numerically efficient way to sample a Wigner distribution for Bogoliubov states was developed by Sinatra *et al.*[37]

## VII. CONCLUSION

We have formally derived all the equations necessary to describe BEC interferometry experiments statistically, given a master equation written in terms of field operators. We have provided general equations required to use the transformation, along with its application to the trapped BEC case. In the latter case, the resulting SDEs can be integrated numerically using conventional methods, and their solutions can be used to calculate all the required observables.

## APPENDIX A: WIRTINGER DIFFERENTIATION

In this paper we are using differentiation of complex functions extensively. Instead of the classical definition of a differential which only works for holomorphic functions, we use Wirtinger differentiation.[60] One can find thorough descriptions of these rules, for example, in Ref. 61; in this section we will only outline the basics.

*Definition A.1.* For a complex variable $z = x + iy$ and a function $f(z) = u(x, y) + iv(x, y)$ the Wirtinger differential is

$$\frac{\partial f(z)}{\partial z} = \frac{1}{2} \left( \frac{\partial f}{\partial x} - i \frac{\partial f}{\partial y} \right).$$

One can easily prove that if $f(z)$ is holomorphic, then the above definition coincides with the classical differential for complex functions. Wirtinger differential obeys sum, product, quotient, and chain differentiation rules (the former is applied as if $f(z) \equiv f(z, z^*)$).

In addition, we will need an area integration over a complex variable:

*Definition A.2.* For a complex variable $z = x + iy$ the integral

$$\int d^2 z \equiv \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} dx \, dy,$$

or, in other words, this stands for a two-dimensional integral over the complex plane.

Such integration has a property similar to a Fourier transformation in real space.

*Lemma A.3. If $\lambda$ is a complex variable, then for any non-negative integers $r$ and $s$:*

$$\int d^2 \alpha \, \alpha^r (\alpha^*)^s \exp(-\lambda \alpha^* + \lambda^* \alpha) = \pi^2 \left( -\frac{\partial}{\partial \lambda^*} \right)^r \left( \frac{\partial}{\partial \lambda} \right)^s \delta(\text{Re}\lambda)\delta(\text{Im}\lambda).$$

*Proof.* First, using known Fourier transform relations, it is easy to prove that for real $x$ and $v$, and non-negative integer $n$

$$\int_{-\infty}^{\infty} dv \, v^n \exp(\pm 2ixv) = \pi(\mp i/2)^n \delta^{(n)}(x).$$

Substituting $\alpha = x + iy$, expanding the $\alpha^r(\alpha^*)^s$ term using binomial theorem and employing the above property, one can reach the statement of the lemma. $\qquad\square$

Another important property is used extensively throughout the paper.

*Lemma A.4. If $f(\lambda, \lambda^*)$ is square-integrable, then for any complex $\alpha$:*

$$\int d^2\lambda \frac{\partial}{\partial \lambda} \left( \exp(-\lambda\alpha^* + \lambda^*\alpha) f(\lambda, \lambda^*) \right) = 0,$$

$$\int d^2\lambda \frac{\partial}{\partial \lambda^*} \left( \exp(-\lambda\alpha^* + \lambda^*\alpha) f(\lambda, \lambda^*) \right) = 0.$$

*Proof.* Square-integrability of f means $\lim\limits_{\mathrm{Re}\lambda \to \infty} f = 0$ and $\lim\limits_{\mathrm{Im}\lambda \to \infty} f = 0$, so the statement of the lemma can be proved by transforming to real variables and integrating. $\qquad\square$

## APPENDIX B: FUNCTIONAL CALCULUS

This section outlines the functional calculus, which is heavily used throughout the paper. A detailed description is given in Ref. 62, and here we only provide some important definitions and results which are used later in the paper. In this section we will use the definitions from Sec. II C, namely, the full basis $\mathbb{B}$ and the restricted basis $\mathbb{M}$. Given the basis, we can define a correspondence between functions of coordinates and their representations in mode space.

*Definition B.1.* Let $\mathbb{F}$ be the space of all functions of coordinates, which consists only of modes from $\mathbb{M}$: $\mathbb{F}_{\mathbb{M}} \equiv (\mathbb{R}^D \to \mathbb{C})_{\mathbb{M}}$ (restricted functions). The composition transformation $\mathcal{C}_{\mathbb{M}} \in \mathbb{C}^{|\mathbb{M}|} \to \mathbb{F}_{\mathbb{M}}$ creates a function from a vector of mode populations,

$$\mathcal{C}_{\mathbb{M}}(\boldsymbol{\alpha}) = \sum_{\boldsymbol{n} \in \mathbb{M}} \phi_{\boldsymbol{n}} \alpha_{\boldsymbol{n}}.$$

The decomposition transformation $\mathcal{C}_{\mathbb{M}}^{-1} \in \mathbb{F} \to \mathbb{C}^{|\mathbb{M}|}$, correspondingly, creates a vector of populations out of a function,

$$(\mathcal{C}_{\mathbb{M}}^{-1}[f])_{\boldsymbol{n}} = \int d\boldsymbol{x} \phi_{\boldsymbol{n}}^* f, \ \boldsymbol{n} \in \mathbb{M}.$$

Note that for any $f \in \mathbb{F}_{\mathbb{M}}$, $\mathcal{C}_{\mathbb{M}}(\mathcal{C}_{\mathbb{M}}^{-1}[f]) \equiv f$.

The result of any nonlinear transformation of a function $f \in \mathbb{F}_{\mathbb{M}}$ is not guaranteed to belong to $\mathbb{F}_{\mathbb{M}}$. This requires explicit projections to be used with other restricted functions. This also applies to the delta function of coordinates. To avoid confusion with the common delta function, we introduce the restricted delta function.

*Definition B.2.* The restricted delta function $\delta_{\mathbb{M}} \in \mathbb{F}_{\mathbb{M}}$ is defined as

$$\delta_{\mathbb{M}}(\boldsymbol{x}', \boldsymbol{x}) = \sum_{\boldsymbol{n} \in \mathbb{M}} \phi_{\boldsymbol{n}}'^* \phi_{\boldsymbol{n}}.$$

Note that $\delta_{\mathbb{M}}^*(\boldsymbol{x}', \boldsymbol{x}) = \delta_{\mathbb{M}}(\boldsymbol{x}, \boldsymbol{x}')$.

Any function can be projected to $\mathbb{M}$ using the projection transformation.

*Definition B.3.* Projection transformation $\mathcal{P}_{\mathbb{M}} \in \mathbb{F} \to \mathbb{F}_{\mathbb{M}}$

$$\mathcal{P}_{\mathbb{M}}[f](\boldsymbol{x}) = (\mathcal{C}_{\mathbb{M}}(\mathcal{C}_{\mathbb{M}}^{-1}[f]))(\boldsymbol{x}) = \sum_{\boldsymbol{n} \in \mathbb{M}} \phi_{\boldsymbol{n}} \int d\boldsymbol{x}' \, \phi_{\boldsymbol{n}}'^* f' = \int d\boldsymbol{x}' \delta_{\mathbb{M}}(\boldsymbol{x}', \boldsymbol{x}) f'.$$

Obviously, $\mathcal{P}_{\mathbb{B}} \equiv \mathbb{1}$. The conjugate of $\mathcal{P}_{\mathbb{M}}$ is thus defined as

$$(\mathcal{P}_{\mathbb{M}}[f](\boldsymbol{x}))^* = \int d\boldsymbol{x}' \delta_{\mathbb{M}}^*(\boldsymbol{x}', \boldsymbol{x}) f'^* = \mathcal{P}_{\mathbb{M}}^*[f^*](\boldsymbol{x}).$$

Let $\mathcal{F}[f] \in \mathbb{F}_{\mathbb{M}} \to \mathbb{F}$ be some transformation (note that the result is not guaranteed to belong to the restricted basis). Because of the bijection between $\mathbb{F}_{\mathbb{M}}$ and $\mathbb{C}^{|\mathbb{M}|}$, $\mathcal{F}$ can be alternatively treated as a function of a vector of complex numbers $\mathcal{F} \in \mathbb{C}^{|\mathbb{M}|} \to \mathbb{C}^\infty$:

$$\mathcal{F}(\boldsymbol{\alpha}) \equiv \mathcal{C}_{\mathbb{M}}^{-1}[\mathcal{F}[\mathcal{C}_{\mathbb{M}}(\boldsymbol{\alpha})]].$$

Using this correspondence, we can define functional differentiation.

*Definition B.4.* The functional derivative $\frac{\delta}{\delta f'} \in (\mathbb{F}_{\mathbb{M}} \to \mathbb{F}) \to (\mathbb{R}^D \to \mathbb{F}_{\mathbb{M}} \to \mathbb{F})$ is defined as

$$\frac{\delta \mathcal{F}[f]}{\delta f'} = \sum_{\boldsymbol{n} \in \mathbb{M}} \phi_{\boldsymbol{n}}'^* \frac{\partial \mathcal{F}(\boldsymbol{\alpha})}{\partial \alpha_{\boldsymbol{n}}}.$$

Note that the transformation being returned differs from the one which was taken: the result of the new transformation is a function of the additional variable from $\mathbb{R}^D$ ($\boldsymbol{x}'$). This variable comes from the function we are differentiating by.

Functional derivatives behave in many ways similar to Wirtinger derivatives. A detailed treatment can be found in Ref. 62. In particular, the following useful lemma gives us the ability to differentiate functionals in a similar way to common functions:

*Lemma B.5. If $g(z)$ is a function of complex variable that can be expanded into series of $z^n(z^*)^m$, and functional $\mathcal{F}[f, f^*] \equiv g(f, f^*)$, $\mathcal{F} \in \mathbb{F}_{\mathbb{M}} \to \mathbb{F}$, then $\delta\mathcal{F}/\delta f'$ and $\delta\mathcal{F}/\delta f'^*$ can be treated as partial differentiation of the functional of two independent variables $f$ and $f^*$. In other words,*

$$\frac{\delta \mathcal{F}}{\delta f'} = \delta_{\mathbb{M}}(\boldsymbol{x}', \boldsymbol{x}) \frac{\partial g(f, f^*)}{\partial f}, \qquad \frac{\delta \mathcal{F}}{\delta f'^*} = \delta_{\mathbb{M}}^*(\boldsymbol{x}', \boldsymbol{x}) \frac{\partial g(f, f^*)}{\partial f^*}.$$

*Definition B.6.* Functional integration $\int \delta^2 f \in (\mathbb{F}_{\mathbb{M}} \to \mathbb{F}) \to \mathbb{C}$ is defined as

$$\int \delta^2 f \, \mathcal{F}[f] = \int d^2\boldsymbol{\alpha} \, \mathcal{F}(\boldsymbol{\alpha}).$$

If the basis contains an infinite number of modes, the integral is treated as a limit $|\mathbb{M}| \to \infty$.

Functional integration has the Fourier-like property analogous to Lemma A.3, but its statement requires the definition of the delta functional:

*Definition B.7.* For a function $\Lambda \in \mathbb{F}_{\mathbb{M}}$ the delta functional is

$$\Delta_{\mathbb{M}}[\Lambda] \equiv \prod_{\boldsymbol{n} \in \mathbb{M}} \delta(\mathrm{Re}\lambda_{\boldsymbol{n}}) \delta(\mathrm{Im}\lambda_{\boldsymbol{n}}),$$

where $\boldsymbol{\lambda} = \mathcal{C}_{\mathbb{M}}^{-1}[\Lambda]$.

The delta functional has the same property as the common delta function:

$$\int \delta^2 \Lambda \mathcal{F}[\Lambda] \Delta_{\mathbb{M}}[\Lambda] = \int d^2\boldsymbol{\lambda} \mathcal{F}(\boldsymbol{\lambda}) \prod_{\boldsymbol{n} \in \mathbb{M}} \delta(\mathrm{Re}\lambda_{\boldsymbol{n}}) \delta(\mathrm{Im}\lambda_{\boldsymbol{n}})$$

$$= \mathcal{F}(\boldsymbol{\lambda})|_{\forall \boldsymbol{n} \in \mathbb{M} \, \lambda_{\boldsymbol{n}} = 0}$$

$$= \mathcal{F}[\Lambda]|_{\Lambda \equiv 0}. \tag{B1}$$

*Lemma B.8 (Functional extension of Lemma A.3). For $\Psi \in \mathbb{F}_{\mathbb{M}}$ and $\Lambda \in \mathbb{F}_{\mathbb{M}}$, and for any non-negative integers r and s:*

$$\int \delta^2 \Psi \, \Psi^r (\Psi^*)^s \exp \left( \int d\boldsymbol{x} \left( -\Lambda \Psi^* + \Lambda^* \Psi \right) \right)$$
$$= \pi^{2|\mathbb{M}|} \left( -\frac{\delta}{\delta \Lambda^*} \right)^r \left( \frac{\delta}{\delta \Lambda} \right)^s \Delta_{\mathbb{M}}[\Lambda].$$

*Proof.* The proof consists of expanding functions into sums of modes and applying Lemma A.3 $|\mathbb{M}|$ times.    □

*Lemma B.9 (Functional extension of Lemma A.4). For a square-integrable functional F*

$$\int \delta^2 \Lambda \frac{\delta}{\delta \Lambda'} \left( D[\Lambda, \Lambda^*, \Psi, \Psi^*] F[\Lambda, \Lambda^*] \right) = 0,$$
$$\int \delta^2 \Lambda \frac{\delta}{\delta \Lambda'^*} \left( D[\Lambda, \Lambda^*, \Psi, \Psi^*] F[\Lambda, \Lambda^*] \right) = 0.$$

*Proof.* Proved by expanding integrals and differentials into modes and applying Lemma A.4.    □

*Lemma B.10. For $\Lambda \in \mathbb{F}_{\mathbb{M}}$ and a bounded functional F*

$$\int \delta^2 \Lambda \frac{\delta}{\delta \Lambda} \left( \left( \left( \frac{\delta}{\delta \Lambda} \right)^s \left( -\frac{\delta}{\delta \Lambda^*} \right)^r \Delta_{\mathbb{M}}[\Lambda] \right) F[\Lambda, \Lambda^*] \right) = 0,$$
$$\int \delta^2 \Lambda \frac{\delta}{\delta \Lambda^*} \left( \left( \left( \frac{\delta}{\delta \Lambda} \right)^s \left( -\frac{\delta}{\delta \Lambda^*} \right)^r \Delta_{\mathbb{M}}[\Lambda] \right) F[\Lambda, \Lambda^*] \right) = 0.$$

*Proof.* Proved by expanding functional integration and differentials into modes and integrating separately over each $\lambda_{\boldsymbol{n}}$, using the fact that any differential of the delta function is zero on the infinity.
□

In order to perform transformations of master equations, we will need a lemma that justifies the "relocation" of the Laplacian (which is a part of the kinetic term in the Hamiltonian) inside the functional integral.

*Lemma B.11. If $\mathcal{F} \in \mathbb{F}_{\mathbb{M}} \to \mathbb{F}$, and $\forall \boldsymbol{n} \in \mathbb{M}$, $\boldsymbol{x} \in \partial A \phi_{\boldsymbol{n}}(\boldsymbol{x}) = 0$, then*

$$\int_A d\boldsymbol{x} \left( \nabla^2 \frac{\delta}{\delta \Psi} \right) \Psi \mathcal{F}[\Psi, \Psi^*] = \int_A d\boldsymbol{x} \frac{\delta}{\delta \Psi} (\nabla^2 \Psi) \mathcal{F}[\Psi, \Psi^*].$$

*Proof.* The proof consists of a function expansion into a mode sum and an application of Green's first identity.    □

Note that the above lemma imposes an additional requirement for basis functions, but in practical applications it is always satisfied. For example, in a plane wave basis eigenfunctions are equal to zero at the border of the bounding box, and in a harmonic oscillator basis they are equal to zero on the infinity (which can be considered the boundary of their integration area). We will assume that this condition is true for any basis we work with.

## APPENDIX C: FUNCTIONAL FOKKER-PLANCK EQUATION

The general approach to numerical solution of the Fokker-Planck equation is to transform it to the equivalent set of SDEs. In the textbooks this transformation is defined for real variables only,[63] while we have functional FPE with complex-valued functions.

Our starting point is the reformulation of the theorem for real-valued multivariable FPE from Ref. [63] in terms of vectors and matrices:

*Lemma C.1 (FPE–SDEs correspondence in convenient form). If $z^T \equiv (z_1 \dots z_M)$ is a set of real-valued variables, Fokker-Planck equation*

$$\frac{dW}{dt} = -\partial_z^T \boldsymbol{a} W + \frac{1}{2} \mathrm{Tr} \left\{ \partial_z \partial_z^T B B^T \right\} W$$

*is equivalent to a set of stochastic differential equations in Itô form*

$$d\boldsymbol{z} = \boldsymbol{a} dt + B d\boldsymbol{Z}$$

*and to a set of stochastic differential equations in Stratonovich form*

$$d\boldsymbol{z} = (\boldsymbol{a} - \boldsymbol{s}) dt + B d\boldsymbol{Z},$$

*where the noise-induced (or spurious) drift vector $\boldsymbol{s}$ has elements*

$$s_i = \frac{1}{2} \sum_{k,j} B_{kj} \frac{\partial}{\partial z_k} B_{ij} = \frac{1}{2} \mathrm{Tr} \left\{ B^T \partial_z \boldsymbol{e}_i^T B \right\},$$

*$\boldsymbol{e}_i$ being the unit vector with elements $(\boldsymbol{e}_i)_j = \delta_{ij}$. Here $W \equiv W(z)$ is a probability distribution, $\boldsymbol{a} \equiv \boldsymbol{a}(z)$ is a vector function, $B \equiv B(z)$ is a matrix function ($B$ having size $M \times L$, where $L$ corresponds to the number of noise sources), $\partial_z^T \equiv (\partial_{z_1} \dots \partial_{z_M})$ is a vector differential, and $\boldsymbol{Z}$ is a standard $L$-dimensional real-valued Wiener process.*

*Proof.* For details see Ref. [63], Secs. 3.3 and 3.4.    □

**Theorem C.2.** *If $\boldsymbol{\alpha}^T \equiv (\alpha_1 \dots \alpha_M)$ is a set of complex-valued variables, Fokker-Planck equation*

$$\frac{dW}{dt} = -\partial_\alpha^T \boldsymbol{a} W - \partial_{\alpha^*}^T \boldsymbol{a}^* W + \mathrm{Tr} \left\{ \partial_{\alpha^*} \partial_\alpha^T B B^H \right\} W$$

*is equivalent to a set of stochastic differential equations in Itô form*

$$d\boldsymbol{\alpha} = \boldsymbol{a} dt + B d\boldsymbol{Z},$$

*or to Stratonovich form*

$$d\boldsymbol{\alpha} = (\boldsymbol{a} - \boldsymbol{s}) dt + B d\boldsymbol{Z},$$

*where noise-induced drift term is*

$$s_j = \frac{1}{2} \mathrm{Tr} \left\{ B^H \partial_{\alpha^*} \boldsymbol{e}_j^T B \right\},$$

*and $\boldsymbol{Z} = (\boldsymbol{X} + i\boldsymbol{Y})/\sqrt{2}$ is a standard $L$-dimensional complex-valued Wiener process, containing two standard real-valued $L$-dimensional Wiener processes $\boldsymbol{X}$ and $\boldsymbol{Y}$.*

*Proof.* Proved straightforwardly by transforming the equation to real variables and applying Lemma C.1.    □

**Theorem C.3.** *If $\boldsymbol{\alpha}^{(j)}$, $j = 1..C$ are C sets of complex variables $\boldsymbol{\alpha}^{(j)} \equiv (\alpha_1^{(j)} \ldots \alpha_{M_j}^{(j)})$, then the Fokker-Planck equation*

$$\frac{dW}{dt} = -\sum_{j=1}^{C} \boldsymbol{\partial}_{\boldsymbol{\alpha}^{(j)}}^T \boldsymbol{a}^{(j)} W - \sum_{j=1}^{C} \boldsymbol{\partial}_{(\boldsymbol{\alpha}^{(j)})^*}^T (\boldsymbol{a}^{(j)})^* W$$

$$+ \sum_{j=1}^{C} \sum_{k=1}^{C} \mathrm{Tr} \left\{ \boldsymbol{\partial}_{(\boldsymbol{\alpha}^{(j)})^*} \boldsymbol{\partial}_{\boldsymbol{\alpha}^{(k)}}^T B^{(k)} (B^{(j)})^H \right\} W$$

*is equivalent to a set of stochastic differential equations in Itô form*

$$d\boldsymbol{\alpha}^{(j)} = \boldsymbol{a}^{(j)} dt + B^{(j)} d\mathbf{Z}, \; j = 1..C$$

*or to Stratonovich form*

$$d\boldsymbol{\alpha}^{(j)} = (\boldsymbol{a}^{(j)} - \boldsymbol{s}^{(j)}) dt + B^{(j)} d\mathbf{Z},$$

*where noise-induced drift term is*

$$s_i^{(j)} = \frac{1}{2} \sum_{k=1}^{C} \mathrm{Tr} \left\{ (B^{(k)})^H \boldsymbol{\partial}_{(\boldsymbol{\alpha}^{(k)})^*} \boldsymbol{e}_i^T B^{(j)} \right\},$$

*and $\mathbf{Z}$ is a standard L-dimensional complex-valued Wiener process.*

    *Proof.* Proved by joining vectors from all components into one vector and applying Theorem C.2.       □

    **Theorem C.4.** *For a probability distribution $W[\boldsymbol{\Psi}, \boldsymbol{\Psi}^*] \in \mathbb{F}_{\mathbb{M}}^C \to \mathbb{R}$, a C-dimensional vector of transformations $\mathcal{A}$ and a $C \times L$ matrix of transformations $\mathcal{B}$ the functional FPE*

$$\frac{dW}{dt} = \int d\boldsymbol{x} \left( -2\mathrm{Re} \left( \boldsymbol{\delta}_{\boldsymbol{\Psi}} \cdot \mathcal{A} \right) + \mathrm{Tr} \left\{ \boldsymbol{\delta}_{\boldsymbol{\Psi}^*} \boldsymbol{\delta}_{\boldsymbol{\Psi}}^T \mathcal{B} \mathcal{B}^H \right\} \right) W$$

*is equivalent to the set of SDEs in Itô form*

$$d\boldsymbol{\Psi} = \mathcal{P} \left[ \mathcal{A} dt + \mathcal{B} d\boldsymbol{Q} \right]$$

*or in Stratonovich form*

$$d\boldsymbol{\Psi} = \mathcal{P} \left[ (\mathcal{A} - \mathcal{S}) dt + \mathcal{B} d\boldsymbol{Q} \right],$$

*where*

$$\mathcal{S}_j = \frac{1}{2} \mathrm{Tr} \left\{ \mathcal{B}^H \boldsymbol{\delta}_{\boldsymbol{\Psi}^*} \boldsymbol{e}_j^T \mathcal{B} \right\},$$

*$\boldsymbol{Q}$ is an L-dimensional vector of standard functional Wiener processes:*

$$Q_l = \sum_{\boldsymbol{n} \in \mathbb{B}} \phi_{\boldsymbol{n}} Z_{l,\boldsymbol{n}}$$

*and $\mathcal{P}^T = \left( \mathcal{P}_{\mathbb{M}_1}, \ldots, \mathcal{P}_{\mathbb{M}_C} \right)$ is a vector of projection transformations.*

*Proof.* Proved by expanding functional derivatives and applying Theorem C.3. The diffusion term has to be transformed in order to conform to the theorem,

$$
\int d\boldsymbol{x}\,\phi_{j,\boldsymbol{m}}\phi_{k,\boldsymbol{n}}^{*}\sum_{l=1}^{L}\mathcal{B}_{kl}\mathcal{B}_{jl}^{*} = \int d\boldsymbol{x}\int d\boldsymbol{x}'\,\phi_{j,\boldsymbol{m}}'\phi_{k,\boldsymbol{n}}^{*}\sum_{l=1}^{L}\mathcal{B}_{jl}'^{*}\mathcal{B}_{kl}\delta(\boldsymbol{x}-\boldsymbol{x}')
$$

$$
= \int d\boldsymbol{x}\int d\boldsymbol{x}'\,\phi_{j,\boldsymbol{m}}'\phi_{k,\boldsymbol{n}}^{*}\sum_{l=1}^{L}\mathcal{B}_{jl}'^{*}\mathcal{B}_{kl}\sum_{\boldsymbol{p}\in\mathbb{B}}\phi_{\boldsymbol{p}}'^{*}\phi_{\boldsymbol{p}}
$$

$$
= \sum_{l=1}^{L}\sum_{\boldsymbol{p}\in\mathbb{B}}\int d\boldsymbol{x}\,\phi_{j,\boldsymbol{m}}\mathcal{B}_{jl}^{*}\phi_{\boldsymbol{p}}^{*}\int d\boldsymbol{x}\,\phi_{k,\boldsymbol{n}}^{*}\mathcal{B}_{kl}\phi_{\boldsymbol{p}}. \tag{C1}
$$

Grouping terms back and recognizing the definition of projection transformation, one gets the statement of the theorem. □

[1] E. P. Wigner, Phys. Rev. **40**, 749–759 (1932).
[2] J. E. Moyal, Math. Proc. Cambridge **45**, 99 (1947).
[3] P. A. M. Dirac, Rev. Mod. Phys. **17**, 195–199 (1945).
[4] P. D. Drummond and A. D. Hardman, Europhys. Lett. **21**, 279–284 (1993).
[5] M. Steel, M. K. Olsen, L. I. Plimak, P. D. Drummond, S. Tan, M. J. Collett, D. F. Walls, and R. Graham, Phys. Rev. A **58**, 4824–4835 (1998).
[6] K. Sakmann, A. I. Streltsov, O. E. Alon, and L. S. Cederbaum, Phys. Rev. Lett. **103**, 220601 (2009).
[7] Y. Li, Y. Castin, and A. Sinatra, Phys. Rev. Lett. **100**, 210401 (2008).
[8] Y. Li, P. Treutlein, J. Reichel, and A. Sinatra, Eur. Phys. J. B **68**, 365–381 (2009).
[9] A. Sinatra, E. Witkowska, J.-C. Dornstetter, Y. Li, and Y. Castin, Phys. Rev. Lett. **107**, 060404 (2011).
[10] P. D. Drummond, R. M. Shelby, S. R. Friberg, and Y. Yamamoto, Nature (London) **365**, 307–313 (1993).
[11] J. F. Corney, J. Heersink, R. Dong, V. Josse, P. D. Drummond, G. Leuchs, and U. L. Andersen, Phys. Rev. A **78**, 023831 (2008).
[12] J. F. Corney, P. D. Drummond, J. Heersink, V. Josse, G. Leuchs, and U. L. Andersen, Phys. Rev. Lett. **97**, 023606 (2006).
[13] L. Isella and J. Ruostekoski, Phys. Rev. A **74**, 063625 (2006).
[14] L. Isella and J. Ruostekoski, Phys. Rev. A **72**, 011601 (2005).
[15] C. Gross, J. Estève, M. K. Oberthaler, A. D. Martin, and J. Ruostekoski, Phys. Rev. A **84**, 011609 (2011).
[16] J. Ruostekoski and L. Isella, Phys. Rev. Lett. **95**, 110403 (2005).
[17] U. Shrestha, J. Javanainen, and J. Ruostekoski, Phys. Rev. A **79**, 043617 (2009).
[18] A. D. Martin and J. Ruostekoski, Phys. Rev. Lett. **104**, 194102 (2010).
[19] A. D. Martin and J. Ruostekoski, New J. Phys. **12**, 055018 (2010).
[20] A. A. Norrie, R. J. Ballagh, and C. W. Gardiner, Phys. Rev. Lett. **94**, 040401 (2005).
[21] A. A. Norrie, R. J. Ballagh, C. W. Gardiner, and A. S. Bradley, Phys. Rev. A **73**, 043618 (2006).
[22] M. Egorov, R. P. Anderson, V. Ivannikov, B. Opanchuk, P. D. Drummond, B. V. Hall, and A. I. Sidorov, Phys. Rev. A **84**, 021605(R) (2011).
[23] B. Opanchuk, M. Egorov, S. E. Hoffmann, A. I. Sidorov, and P. D. Drummond, Europhys. Lett. **97**, 50003 (2012).
[24] B. Opanchuk, Q.-Y. He, M. D. Reid, and P. D. Drummond, Phys. Rev. A **86**, 023625 (2012).
[25] P. Deuar and P. D. Drummond, Phys. Rev. Lett. **98**, 120402 (2007).
[26] P. D. Drummond and C. W. Gardiner, J. Phys. A **13**, 2353–2368 (1980).
[27] S. Chaturvedi, K. Dechoum, and P. Drummond, Phys. Rev. A **65**, 033805 (2002).
[28] K. Dechoum, P. D. Drummond, S. Chaturvedi, and M. D. Reid, Phys. Rev. A **70**, 053807 (2004).
[29] P. Drummond and P. Kinsler, Phys. Rev. A **40**, 4813–4816 (1989).
[30] P. Kinsler and P. Drummond, Phys. Rev. A **43**, 6194–6208 (1991).
[31] Q.-Y. He, M. D. Reid, B. Opanchuk, R. Polkinghorne, L. E. C. Rosales-Zárate, and P. D. Drummond, Fron. Phys. **7**, 16–30 (2012).
[32] R. Graham and H. Haken, Z. Phys. **234**, 193–206 (1970).
[33] R. Graham and H. Haken, Z. Phys. **235**, 166–180 (1970).
[34] C. W. Gardiner and M. J. Davis, J. Phys. B **36**, 4731–4753 (2003).
[35] P. B. Blakie, A. S. Bradley, M. J. Davis, R. J. Ballagh, and C. W. Gardiner, Adv. Phys. **57**, 363–455 (2008).
[36] A. Polkovnikov, Ann. Phys. **325**, 1790–1852 (2010).
[37] A. Sinatra, C. Lobo, and Y. Castin, J. Phys. B **35**, 3599–3631 (2002).
[38] S. Chaturvedi and P. Drummond, Eur. Phys. J. B **8**, 251–267 (1999).
[39] S. Chaturvedi, P. Drummond, and D. F. Walls, J. Phys. A **10**, L187–L192 (1977).
[40] M. Reid and D. Walls, Phys. Rev. A **33**, 4465–4468 (1986).
[41] P. Rabl, A. Shnirman, and P. Zoller, Phys. Rev. B **70**, 205304 (2004).
[42] P. Drummond and I. Mortimer, J. Comput. Phys. **89**, 252 (1990).
[43] M. J. Werner and P. D. Drummond, J. Comput. Phys. **132**, 312–326 (1997).
[44] J. Wilkie and M. Cetinbas, Phys. Lett. A **337**, 166–182 (2005).

[45] G. Collecutt and P. Drummond, Comput. Phys. Commun. **142**, 219–223 (2001).

[46] G. R. Dennis, J. J. Hope, and M. T. Johnsson, Comput. Phys. Commun. **184**, 201–208 (2013).

[47] S. A. Morgan, J. Phys. B **33**, 3847–3893 (2000).

[48] S. Kokkelmans, J. N. Milstein, M. L. Chiofalo, R. Walser, and M. J. Holland, Phys. Rev. A **65**, 053617 (2002).

[49] M. W. Jack, Phys. Rev. Lett. **89**, 140402 (2002).

[50] H. Weyl, *The Theory of Groups and Quantum Mechanics* (Dover Publications, 1950), p. 444.

[51] K. Cahill and R. Glauber, Phys. Rev. **177**, 1857–1881 (1969).

[52] M. Hillery, R. F. O'Connell, M. O. Scully, and E. P. Wigner, Phys. Rep. **106**, 121–167 (1984).

[53] C. W. Gardiner and P. Zoller, *Quantum Noise*, 4th ed. (Springer, 2004), p. 449.

[54] W. H. Louisell, *Quantum Statistical Properties of Radiation* (John Wiley & Sons, Inc., 1990), p. 544.

[55] A. Polkovnikov, Phys. Rev. A **68**, 053604 (2003).

[56] A. A. Norrie, R. J. Ballagh, and C. W. Gardiner, Phys. Rev. A **73**, 043617 (2006).

[57] J. Javanainen and J. Ruostekoski, New J. Phys. **15**, 013005 (2013).

[58] C. W. Gardiner, *Handbook of Stochastic Methods*, 2nd ed. (Springer-Verlag, Berlin, 1997), p. 442.

[59] J. Ruostekoski and A. D. Martin, "Truncated Wigner method for Bose gases," preprint arXiv:1009.1073 (2010).

[60] W. Wirtinger, Math. Ann. **97**, 357–375 (1927).

[61] K. Kreutz-Delgado, "The complex gradient operator and the CR-calculus," preprint arXiv:0906.4835 (2009).

[62] B. J. Dalton, "Phase space theory of Bose-Einstein condensates and time-dependent modes," preprint arXiv:1108.1251 (2011).

[63] H. Risken, *The Fokker-Planck Equation*, 2nd ed. (Springer-Verlag, Berlin, 1996).